Non-Parametric Parametricity
(Technical Appendix)

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1.1 Syntax and Semantics

Syntax

(variables)	α	$\in$	TVar
(types)	τ	$::=$	$\alpha \mid b \mid \tau \times \tau \mid \tau \to \tau \mid \forall\alpha.\tau \mid \exists\alpha.\tau$
(values)	v	$::=$	$x \mid c \mid \langle v, v\rangle \mid \lambda x{:}\tau.e \mid \Lambda\alpha.e \mid \mathsf{pack}\ \langle\tau, v\rangle\ \mathsf{as}\ \tau$
(expressions)	e	$::=$	$v \mid \langle e, e\rangle \mid e.1 \mid e.2 \mid e\, e \mid e\, \tau \mid \mathsf{pack}\ \langle\tau, v\rangle\ \mathsf{as}\ \tau \mid$ $\mathsf{unpack}\ \langle\alpha, x\rangle{=}e\ \mathsf{in}\ e \mid \mathsf{new}\ \alpha{\approx}\tau\ \mathsf{in}\ e \mid\ \mathsf{cast}\ \tau\, \tau$
(contexts)	C	$::=$	$[\_] \mid \langle C, e\rangle \mid \langle e, C\rangle \mid C.1 \mid C.2 \mid \lambda x{:}\tau.C \mid C\, e \mid e\, C \mid$ $\Lambda\alpha.C \mid C\,\tau \mid\ \mathsf{pack}\langle\tau, C\rangle \mid\ \mathsf{unpack}\langle\alpha, x\rangle{=}C\ \mathsf{in}\ e \mid$ $\mathsf{unpack}\langle\alpha, x\rangle{=}e\ \mathsf{in}\ C \mid \mathsf{new}\ \alpha{\approx}\tau\ \mathsf{in}\ C$
(stores)	σ	$::=$	$\epsilon \mid \sigma, \alpha{\approx}\tau$
(type environments)	Δ	$::=$	$\epsilon \mid \Delta, \alpha \mid \Delta, \alpha{\approx}\tau$
(value environments)	Γ	$::=$	$\epsilon \mid \Gamma, x{:}\tau$
(type substitutions)	δ	$::=$	$\emptyset \mid \delta, \alpha{\mapsto}\tau$
(value substitutions)	γ	$::=$	$\emptyset \mid \gamma, x{\mapsto}v$

Reduction

$\boxed{\sigma; e \hookrightarrow \sigma; e}$

$$(\textsc{Rproj1.1})\ \frac{}{\sigma; \langle v_1, v_2\rangle.1 \hookrightarrow \sigma; v_1} \qquad (\textsc{Rproj2.1})\ \frac{}{\sigma; \langle v_1, v_2\rangle.2 \hookrightarrow \sigma; v_2}$$

$$(\textsc{Rapp})\ \frac{}{\sigma; (\lambda x{:}\tau.e)\, v \hookrightarrow \sigma; e[v/x]} \qquad (\textsc{Rinst})\ \frac{}{\sigma; (\Lambda\alpha.e)\, \tau \hookrightarrow \sigma; e[\tau/\alpha]}$$

$$(\textsc{Runpack})\ \frac{}{\sigma; \mathsf{unpack}\langle\alpha, x\rangle{=}(\mathsf{pack}\langle\tau, v\rangle\ \mathsf{as}\ \tau')\ \mathsf{in}\ e \hookrightarrow \sigma; e[\tau/\alpha][v/x]}$$

$$(\textsc{Rpair.1})\ \frac{\sigma; e_1 \hookrightarrow \sigma'; e_1'}{\sigma; \langle e_1, e_2\rangle \hookrightarrow \sigma'; \langle e_1', e_2\rangle} \qquad (\textsc{Rpair.2})\ \frac{\sigma; e_2 \hookrightarrow \sigma'; e_2'}{\sigma; \langle e_1, e_2\rangle \hookrightarrow \sigma'; \langle e_1, e_2'\rangle}$$

$$(\textsc{Rproj1.2})\ \frac{\sigma; e \hookrightarrow \sigma'; e'}{\sigma; e.1 \hookrightarrow \sigma'; e'.1} \qquad (\textsc{Rproj2.2})\ \frac{\sigma; e \hookrightarrow \sigma'; e'}{\sigma; e.2 \hookrightarrow \sigma'; e'.2}$$

$$(\textsc{Rapp.1})\ \frac{\sigma; e_1 \hookrightarrow \sigma'; e_1'}{\sigma; e_1\, e_2 \hookrightarrow \sigma'; e_1'\, e_2} \qquad (\textsc{Rapp.2})\ \frac{\sigma; e_2 \hookrightarrow \sigma'; e_2'}{\sigma; v\, e_2 \hookrightarrow \sigma'; v\, e_2'}$$

$$(\textsc{Rinst.1})\ \frac{\sigma; e \hookrightarrow \sigma'; e'}{\sigma; e\, \tau \hookrightarrow \sigma'; e'\, \tau} \qquad (\textsc{Rpack.1})\ \frac{\sigma; e \hookrightarrow \sigma'; e'}{\sigma; \mathsf{pack}\langle\tau, e\rangle\ \mathsf{as}\ \tau' \hookrightarrow \sigma'; \mathsf{pack}\langle\tau, e'\rangle\ \mathsf{as}\ \tau'}$$

$$(\textsc{Runpack.1})\ \frac{\sigma; e_1 \hookrightarrow \sigma'; e_1'}{\sigma; \mathsf{unpack}\langle\alpha, x\rangle{=}e_1\ \mathsf{in}\ e_2 \hookrightarrow \sigma'; \mathsf{unpack}\langle\alpha, x\rangle{=}e_1'\ \mathsf{in}\ e_2}$$

$$(\textsc{Rnew})\ \frac{\alpha' \notin \mathrm{dom}(\sigma)}{\sigma; \mathsf{new}\ \alpha{\approx}\tau\ \mathsf{in}\ e \hookrightarrow \sigma, \alpha'{\approx}\tau; e[\alpha'/\alpha]}$$

$$(\textsc{Rcast.1})\frac{}{\sigma; \mathsf{cast}\ \tau\ \tau \hookrightarrow \sigma; \lambda x_1{:}\tau.\lambda x_2{:}\tau.x_1}$$

$$(\textsc{Rcast.2})\frac{\tau_1 \neq \tau_2}{\sigma; \mathsf{cast}\ \tau_1\ \tau_2 \hookrightarrow \lambda x_1{:}\tau_1.\lambda x_2{:}\tau_2.x_2}$$

Type Environments

$\boxed{\vdash \Delta}$

$$\frac{}{\vdash \epsilon} \qquad \frac{\vdash \Delta \qquad \alpha \notin \mathrm{dom}(\Delta)}{\vdash \Delta, \alpha} \qquad \frac{\Delta \vdash \tau \qquad \alpha \notin \mathrm{dom}(\Delta)}{\vdash \Delta, \alpha{\approx}\tau}$$

Value Environments

$\boxed{\Delta \vdash \Gamma}$

$$\frac{\vdash \Delta}{\Delta \vdash \epsilon} \qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash \tau \qquad x \notin \mathrm{dom}(\Gamma)}{\Delta \vdash \Gamma, x{:}\tau}$$

Types

$\boxed{\Delta \vdash \tau}$

$$(\textsc{Tvar})\frac{\vdash \Delta \qquad \alpha \in \Delta}{\Delta \vdash \alpha} \qquad (\textsc{Tname})\frac{\vdash \Delta \qquad \alpha{\approx}\tau \in \Delta}{\Delta \vdash \alpha}$$

$$(\textsc{Tbase})\frac{\vdash \Delta}{\Delta \vdash b} \qquad (\textsc{Ttimes})\frac{\Delta \vdash \tau_1 \qquad \Delta \vdash \tau_2}{\Delta \vdash \tau_1 \times \tau_2} \qquad (\textsc{Tarr})\frac{\Delta \vdash \tau_1 \qquad \Delta \vdash \tau_2}{\Delta \vdash \tau_1 \to \tau_2}$$

$$(\textsc{Tall})\frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \forall \alpha.\tau} \qquad (\textsc{Texists})\frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \exists \alpha.\tau}$$

Type Compatibility

$\boxed{\Delta \vdash \tau \approx \tau'}$

$$(\textsc{Qvar})\frac{\vdash \Delta \qquad \alpha \in \Delta}{\Delta \vdash \alpha \approx \alpha} \qquad (\textsc{Qname})\frac{\vdash \Delta \qquad \alpha{\approx}\tau \in \Delta}{\Delta \vdash \alpha \approx \tau} \qquad (\textsc{Qbase})\frac{\vdash \Delta}{\Delta \vdash b \approx b}$$

$$(\textsc{Qtimes})\frac{\Delta \vdash \tau_1 \approx \tau_1' \qquad \Delta \vdash \tau_2 \approx \tau_2'}{\Delta \vdash \tau_1 \times \tau_2 \approx \tau_1' \times \tau_2'} \qquad (\textsc{Qarr})\frac{\Delta \vdash \tau_1 \approx \tau_1' \qquad \Delta \vdash \tau_2 \approx \tau_2'}{\Delta \vdash \tau_1 \to \tau_2 \approx \tau_1' \to \tau_2'}$$

$$(\textsc{Qall})\frac{\Delta, \alpha \vdash \tau \approx \tau'}{\Delta \vdash \forall \alpha.\tau \approx \forall \alpha.\tau'} \qquad (\textsc{Qexists})\frac{\Delta, \alpha \vdash \tau \approx \tau'}{\Delta \vdash \exists \alpha.\tau \approx \exists \alpha.\tau'}$$

$$(\textsc{Qsym})\frac{\Delta \vdash \tau' \approx \tau}{\Delta \vdash \tau \approx \tau'} \qquad (\textsc{Qtrans})\frac{\Delta \vdash \tau \approx \tau'' \qquad \Delta \vdash \tau'' \approx \tau'}{\Delta \vdash \tau \approx \tau'}$$

Expressions

$\boxed{\Delta; \Gamma \vdash e : \tau}$

$$(\textsc{Evar})\frac{\Delta \vdash \Gamma \qquad x{:}\tau \in \Gamma}{\Delta; \Gamma \vdash x : \tau} \qquad (\textsc{Econ})\frac{\Delta \vdash \Gamma}{\Delta; \Gamma \vdash c : b}$$

$$(\textsc{Epair})\frac{\Delta; \Gamma \vdash e_1 : \tau_1 \qquad \Delta; \Gamma \vdash e_2 : \tau_2}{\Delta; \Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \qquad (\textsc{Eproj})\frac{\Delta; \Gamma \vdash e : \tau_1 \times \tau_2}{\Delta; \Gamma \vdash e.i : \tau_i}$$

$$(\textsc{Eabs})\frac{\Delta; \Gamma, x{:}\tau_1 \vdash e : \tau_2}{\Delta; \Gamma \vdash \lambda x{:}\tau_1.e : \tau_1 \to \tau_2} \qquad (\textsc{Eapp})\frac{\Delta; \Gamma \vdash e_1 : \tau_2 \to \tau \qquad \Delta; \Gamma \vdash e_2 : \tau_2}{\Delta; \Gamma \vdash e_1\, e_2 : \tau}$$

$$(\textsc{Egen})\frac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.e : \forall\alpha.\tau} \qquad (\textsc{Einst})\frac{\Delta; \Gamma \vdash e : \forall\alpha.\tau \qquad \Delta \vdash \tau_2}{\Delta; \Gamma \vdash e\, \tau_2 : \tau[\tau_2/\alpha]}$$

$$(\textsc{Epack})\frac{\Delta; \Gamma \vdash e : \tau[\tau_1/\alpha] \qquad \Delta \vdash \tau_1}{\Delta; \Gamma \vdash \mathsf{pack}\langle \tau_1, e \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau}$$

$$(\textsc{Eunpack})\frac{\Delta; \Gamma \vdash e_1 : \exists\alpha.\tau_1 \qquad \Delta, \alpha; \Gamma, x{:}\tau_1 \vdash e_2 : \tau \qquad \Delta \vdash \tau}{\Delta; \Gamma \vdash \mathsf{unpack}\langle \alpha, x \rangle{=}e_1\ \mathsf{in}\ e_2 : \tau}$$

$$(\textsc{Enew})\frac{\Delta, \alpha{\approx}\tau'; \Gamma \vdash e : \tau \qquad \Delta \vdash \tau \qquad \Delta \vdash \Gamma}{\Delta; \Gamma \vdash \mathsf{new}\ \alpha{\approx}\tau'\ \mathsf{in}\ e : \tau} \qquad (\textsc{Ecast})\frac{\Delta \vdash \Gamma \qquad \Delta \vdash \tau_1 \qquad \Delta \vdash \tau_2}{\Delta; \Gamma \vdash \mathsf{cast}\ \tau_1\, \tau_2 : \tau_1 \to \tau_2 \to \tau_2}$$

$$(\textsc{Econv})\frac{\Delta; \Gamma \vdash e : \tau' \qquad \Delta \vdash \tau \approx \tau'}{\Delta; \Gamma \vdash e : \tau}$$

Configurations

$\boxed{\vdash \sigma; e : \tau}$

$$\frac{\Delta = \sigma \qquad \Delta; \epsilon \vdash e : \tau \qquad \epsilon \vdash \tau}{\vdash \sigma; e : \tau}$$

Contexts

$\boxed{C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta; \Gamma; \tau)}$

$$(\textsc{Cempty})\frac{\Delta' \supseteq \Delta \quad \Gamma' \supseteq \Gamma \quad \Delta' \vdash \Gamma'}{[\_] : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau)}$$

$$(\textsc{Cabs})\frac{C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma', x{:}\tau_1; \tau_2)}{\lambda x{:}\tau_1.C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_1 \to \tau_2)}$$

$$(\textsc{Cpair.1})\frac{C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_1) \qquad \Delta'; \Gamma' \vdash e : \tau_2}{\langle C, e \rangle : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_1 \times \tau_2)}$$

$$(\textsc{Cpair.2})\frac{C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_2) \qquad \Delta'; \Gamma' \vdash e : \tau_1}{\langle e, C \rangle : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_1 \times \tau_2)}$$

$$(\textsc{Cproj})\frac{C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_1 \times \tau_2)}{C.i : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau_i)}$$

$$(\textsc{CApp.1})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_1 \to \tau_2) \qquad \Delta';\Gamma' \vdash e : \tau_1}{C\,e : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_2)}$$

$$(\textsc{CApp.2})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_1) \qquad \Delta';\Gamma' \vdash e : \tau_1 \to \tau_2}{e\,C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_2)}$$

$$(\textsc{CGen})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta',\alpha;\Gamma';\tau')}{\Lambda\alpha.C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\forall\alpha.\tau')}$$

$$(\textsc{CInst})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\forall\alpha.\tau_1) \qquad \Delta' \vdash \tau_2}{C\,\tau_2 : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_1[\tau_2/\alpha])}$$

$$(\textsc{CPack})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_1[\tau_2/\alpha]) \qquad \Delta' \vdash \tau_2}{\mathsf{pack}\langle\tau_2, C\rangle : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\exists\alpha.\tau_1)}$$

$$(\textsc{CUnpack.1})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\exists\alpha.\tau_1) \qquad \Delta',\alpha;\Gamma',x{:}\tau_1 \vdash e_2 : \tau_2 \qquad \Delta' \vdash \tau_2}{\mathsf{unpack}\langle\alpha,x\rangle{=}C\ \mathsf{in}\ e : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_2)}$$

$$(\textsc{CUnpack.2})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta',\alpha;\Gamma',x{:}\tau_1;\tau_2) \qquad \Delta';\Gamma' \vdash e : \exists\alpha.\tau_1 \qquad \Delta' \vdash \tau_2}{\mathsf{unpack}\langle\alpha,x\rangle{=}e\ \mathsf{in}\ C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_2)}$$

$$(\textsc{CNew})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta',\alpha{\approx}\tau_1;\Gamma';\tau_2) \qquad \Delta' \vdash \tau_2 \qquad \Delta' \vdash \Gamma'}{\mathsf{new}\ \alpha{\approx}\tau'\ \mathsf{in}\ C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau_2)}$$

$$(\textsc{CConv})\frac{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau') \qquad \Delta' \vdash \tau' \approx \tau''}{C : (\Delta;\Gamma;\tau) \rightsquigarrow (\Delta';\Gamma';\tau'')}$$

Type Substitutions $\boxed{\Delta \vdash \delta : \Delta}$

$$\frac{\vdash \Delta'}{\Delta' \vdash \emptyset : \epsilon} \qquad \frac{\Delta' \vdash \delta : \Delta \qquad \Delta' \vdash \tau}{\Delta' \vdash \delta, \alpha{\mapsto}\tau : \Delta, \alpha} \qquad \frac{\Delta' \vdash \delta : \Delta \qquad \Delta' \vdash \tau' \approx \tau}{\Delta' \vdash \delta, \alpha{\mapsto}\tau' : \Delta, \alpha{\approx}\tau}$$

Value Substitutions $\boxed{\Delta;\Gamma \vdash \gamma : \Gamma}$

$$\frac{\Delta \vdash \Gamma'}{\Delta;\Gamma' \vdash \emptyset : \epsilon} \qquad \frac{\Delta;\Gamma' \vdash \gamma : \Gamma \qquad \Delta;\Gamma' \vdash v : \tau}{\Delta;\Gamma' \vdash \gamma, x{\mapsto}v : \Gamma, x{:}\tau}$$

1.2 Structural Properties

Lemma 1.1 (Weakening)

1. If $\Delta \vdash \tau$ and $\Delta' \supseteq \Delta$ and $\vdash \Delta'$, then $\Delta' \vdash \tau$.
2. If $\Delta \vdash \Gamma$ and $\Delta' \supseteq \Delta$ and $\vdash \Delta'$, then $\Delta' \vdash \Gamma$.
3. If $\Delta;\Gamma \vdash e : \tau$ and $\Delta' \supseteq \Delta$ and $\vdash \Delta'$, then $\Delta';\Gamma \vdash e : \tau$.
4. If $\Delta;\Gamma \vdash e : \tau$ and $\Gamma' \supseteq \Gamma$ and $\Delta \vdash \Gamma'$, then $\Delta;\Gamma' \vdash e : \tau$.

Proof: Each by induction on the first derivation. ■

Lemma 1.2 (Substitution)

1. If $\Delta \vdash \tau$ and $\Delta' \vdash \delta : \Delta$, then $\Delta' \vdash \delta(\tau)$.
2. If $\Delta \vdash \Gamma$ and $\Delta' \vdash \delta : \Delta$, then $\Delta' \vdash \delta(\Gamma)$.
3. If $\Delta; \Gamma \vdash e : \tau$ and $\Delta' \vdash \delta : \Delta$, then $\Delta'; \delta(\Gamma) \vdash \delta(e) : \delta(\tau)$.
4. If $\Delta; \Gamma \vdash e : \tau$ and $\Delta; \Gamma' \vdash \gamma : \Gamma$, then $\Delta; \Gamma' \vdash \gamma(e) : \tau$.

Proof: Each by induction on the first derivation. ■

Lemma 1.3 (Validity)

1. If $\Delta \vdash \tau$, then $\vdash \Delta$.
2. If $\Delta \vdash \Gamma$, then $\vdash \Delta$.
3. If $\Delta; \Gamma \vdash e : \tau$, then $\vdash \Delta$ and $\Delta \vdash \Gamma$ and $\Delta \vdash \tau$.

Proof: Each by induction on the derivation. ■

Lemma 1.4 (Variable Containment)

1. If $\Delta \vdash \tau$ and $\alpha \in \mathrm{ftv}(\tau)$, then $\alpha \in \Delta$.
2. If $\Delta \vdash \Gamma$ and $\alpha \in \mathrm{ftv}(\Gamma)$, then $\alpha \in \Delta$.
3. If $\Delta; \Gamma \vdash e : \tau$ and $\alpha \in \mathrm{ftv}(\Gamma) \cup \mathrm{ftv}(e) \cup \mathrm{ftv}(\tau)$, then $\alpha \in \Delta$.
4. If $\Delta; \Gamma \vdash e : \tau$ and $x \in \mathrm{fvv}(e)$, then $x \in \mathrm{dom}(\Gamma)$.

Proof: Each by induction on the derivation. ■

1.3 Soundness

Lemma 1.5 (Preservation)
If $\sigma; e \hookrightarrow \sigma'; e'$ and $\vdash \sigma; e : \tau$, then $\vdash \sigma'; e' : \tau$.

Proof: By induction on the first derivation. ■

Lemma 1.6 (Canonical Values)
Assume $\vdash \sigma; v : \tau$. Then:

1. If $\tau = \tau_1 \times \tau_2$, then $v = \langle v_1, v_2 \rangle$.
2. If $\tau = \tau_1 \rightarrow \tau_2$, then $v = \lambda x{:}\tau_1.e$.
3. If $\tau = \forall \alpha.\tau_1$, then $v = \Lambda \alpha.e$.
4. If $\tau = \exists \alpha.\tau_1$, then $v = \mathsf{pack}\ \langle \tau_2, v_1 \rangle\ \mathsf{as}\ \tau$.

Proof: By induction on the derivation. ■

Lemma 1.7 (Progress)
If $\vdash \sigma; e : \tau$ and $e \neq v$, then $\sigma; e \hookrightarrow \sigma'; e'$.

Proof: By induction on the derivation. ■

1.4 Contextual Approximation

$$\begin{array}{lcl} \Delta; \Gamma \vdash e_1 \preceq e_2 : \tau & \overset{\text{def}}{\iff} & \Delta; \Gamma \vdash e_1 : \tau \wedge \Delta; \Gamma \vdash e_2 : \tau \wedge \\ & & \forall \vdash \sigma. \forall C : (\Delta; \Gamma; \tau) \rightsquigarrow (\sigma; \epsilon; \tau').\ \sigma; C[e_1]\downarrow \implies \sigma; C[e_2]\downarrow \end{array}$$

2 Parameterized Logical Relation for G

2.1 Minor Differences Between Paper and Appendix

There are some minor differences between the definitions in this appendix and the ones in the paper.

1. In the paper, one logical relation is presented after the other. Here we present all four relations at once, combined into a unified definition that uses a parameter ι ranging over $\bullet$ (the *proper* relation), $\circ$, $+$, and $-$.

2. In the definition of the logical relation, the paper uses the $\triangleright$ operator to make it clear where we need to go down a step. Here we are still using an explicit quantification over future worlds in those places, which is equivalent in the end but has the disadvantage that it is not immediately obvious to the reader whether a particular quantification is needed to maintain monotonicity or to maintain well-foundedness.

3. In the definition of World and $T[\![\Omega]\!]$ (and thus in many of the proofs), the appendix still uses $\sigma^*(\tau)$ (defined below) to normalise a type τ with respect to a store σ. The paper instead indirectly, i.e., without using the notation σ^*, requires the existence of a type τ' that equals $\sigma^*(\tau)$ and then works with τ'.

2.2 Definition

$$
\begin{aligned}
\epsilon^* &\stackrel{\text{def}}{=} \emptyset \\
(\sigma, \alpha)^* &\stackrel{\text{def}}{=} \sigma^* \\
(\sigma, \alpha{\approx}\tau)^* &\stackrel{\text{def}}{=} \sigma^*, \alpha{\mapsto}\sigma^*(\tau)
\end{aligned}
$$

$$
\begin{array}{lll}
\mathrm{Typ}_A & \stackrel{\text{def}}{=} & \{\tau \mid \mathrm{fv}(\tau) \subseteq A\} \\
\\
\mathrm{Atom}_n[\tau_1, \tau_2] & \stackrel{\text{def}}{=} & \{(k, w, e_1, e_2) \mid k < n \wedge w \in \mathrm{World}_k \wedge \vdash w.\sigma_1; e_1 : \tau_1 \wedge \vdash w.\sigma_2; e_2 : \tau_2\} \\
\mathrm{Rel}_n[\tau_1, \tau_2] & \stackrel{\text{def}}{=} & \{R \subseteq \mathrm{Atom}^{\mathrm{val}}_n[\tau_1, \tau_2] \mid \forall (k, w, v_1, v_2) \in R, (k', w') \sqsupseteq (k, w).\ (k', w', v_1, v_2) \in R\} \\
\mathrm{SomeRel}_n & \stackrel{\text{def}}{=} & \{(\tau_1, \tau_2, R) \mid \tau_1 \in \mathrm{Typ}_\emptyset \wedge \tau_2 \in \mathrm{Typ}_\emptyset \wedge R \in \mathrm{Rel}_n[\tau_1, \tau_2]\} \\
\mathrm{Interp}_n & \stackrel{\text{def}}{=} & \{\rho \in \mathrm{TVar} \stackrel{\text{fin}}{\to} \mathrm{SomeRel}_n\} \\
\mathrm{Conc} & \stackrel{\text{def}}{=} & \{\eta \in \mathrm{TVar} \stackrel{\text{fin}}{\to} \mathrm{TVar} \times \mathrm{TVar} \mid \forall \alpha, \alpha' \in \mathrm{dom}(\eta).\ \alpha \neq \alpha' \implies \\
& & \quad \eta^1(\alpha) \neq \eta^1(\alpha') \wedge \eta^2(\alpha) \neq \eta^2(\alpha')\} \\
\mathrm{World}_n & \stackrel{\text{def}}{=} & \{(\sigma_1, \sigma_2, \eta, \rho) \mid \vdash \sigma_1 \wedge \vdash \sigma_2 \wedge \eta \in \mathrm{Conc} \wedge \rho \in \mathrm{Interp}_n \wedge \\
& & \quad \mathrm{dom}(\eta) = \mathrm{dom}(\rho) \wedge \rho^1 = \sigma_1^* \cdot \eta^1 \wedge \rho^2 = \sigma_2^* \cdot \eta^2\} \\
\mathrm{Interp} & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} \mathrm{Interp}_n \\
\mathrm{World} & \stackrel{\text{def}}{=} & \bigcup_{n \geq 0} \mathrm{World}_n
\end{array}
$$

$$
\begin{aligned}
\lfloor R \rfloor_n &\stackrel{\text{def}}{=} \{(k, w, v_1, v_2) \mid k < n \wedge (k, w, v_1, v_2) \in R\} \\
\lfloor (\tau_1, \tau_2, R) \rfloor_n &\stackrel{\text{def}}{=} (\tau_1, \tau_2, \lfloor R \rfloor_n) \\
\lfloor \rho \rfloor_n &\stackrel{\text{def}}{=} \{\alpha \mapsto \lfloor r \rfloor_n \mid \rho(\alpha) = r\} \\
\lfloor (\sigma_1, \sigma_2, \eta, \rho) \rfloor_n &\stackrel{\text{def}}{=} (\sigma_1, \sigma_2, \eta, \lfloor \rho \rfloor_n)
\end{aligned}
$$

$$
\begin{array}{lcl}
\eta' \sqsupseteq \eta & \stackrel{\text{def}}{\iff} & \forall \alpha \in \text{dom}(\eta).\ \eta(\alpha) = \eta'(\alpha) \\
\rho' \sqsupseteq \rho & \stackrel{\text{def}}{\iff} & \forall \alpha \in \text{dom}(\rho).\ \rho(\alpha) = \rho'(\alpha) \\
(k', w') \sqsupseteq (k, w) & \stackrel{\text{def}}{\iff} & k' \leq k \wedge w' \in \text{World}_{k'} \wedge w'.\sigma_1 \supseteq w.\sigma_1 \wedge w'.\sigma_2 \supseteq w.\sigma_2 \wedge \\
 & & w'.\eta \sqsupseteq w.\eta \wedge w'.\rho \sqsupseteq \lfloor w.\rho \rfloor_{k'} \wedge \\
 & & \text{rng}(w'.\eta^i) \setminus \text{rng}(w.\eta^i) \subseteq \text{dom}(w'.\sigma_i) \setminus \text{dom}(w.\sigma_i) \\
(k', w') \sqsupset (k, w) & \stackrel{\text{def}}{\iff} & k' < k \wedge (k', w') \sqsupseteq (k, w)
\end{array}
$$

$$
\iota \ ::= \ \bullet \mid \ - \ \mid \ + \ \mid \circ
$$

$$
\begin{array}{ll}
\neg\bullet \stackrel{\text{def}}{=} \bullet & \quad |\bullet| \stackrel{\text{def}}{=} \bullet \\
\neg- \stackrel{\text{def}}{=} + & \quad |-| \stackrel{\text{def}}{=} \bullet \\
\neg+ \stackrel{\text{def}}{=} - & \quad |+| \stackrel{\text{def}}{=} \circ \\
\neg\circ \stackrel{\text{def}}{=} \circ & \quad |\circ| \stackrel{\text{def}}{=} \circ
\end{array}
$$

$$
\begin{array}{lcl}
V^\iota_n[\![\alpha]\!]\rho & \stackrel{\text{def}}{=} & \lfloor \rho(\alpha).R \rfloor_n \\
V^\iota_n[\![b]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, c, c) \in \text{Atom}_n[b, b]\} \\
V^\iota_n[\![\tau \times \tau']\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, \langle v_1, v_1' \rangle, \langle v_2, v_2' \rangle) \in \text{Atom}_n[\rho^1(\tau \times \tau'), \rho^2(\tau \times \tau')] \mid \\
& & \quad (k, w, v_1, v_2) \in V^\iota_n[\![\tau]\!]\rho \wedge (k, w, v_1', v_2') \in V^\iota_n[\![\tau']\!]\rho\} \\
V^\iota_n[\![\tau' \to \tau]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, \lambda x{:}\tau_1.e_1, \lambda x{:}\tau_2.e_2) \in \text{Atom}_n[\rho^1(\tau' \to \tau), \rho^2(\tau' \to \tau)] \mid \\
& & \quad \forall (k', w', v_1, v_2) \in V^{\neg\iota}_n[\![\tau']\!]\rho. \\
& & \quad (k', w') \sqsupseteq (k, w) \implies (k', w', e_1[v_1/x], e_2[v_2/x]) \in E^\iota_n[\![\tau]\!]\rho\} \\
V^\iota_n[\![\forall \alpha.\tau]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in \text{Atom}_n[\rho^1(\forall\alpha.\tau), \rho^2(\forall\alpha.\tau)] \mid \\
& & \quad \forall (k'', w'') \sqsupset (k', w') \sqsupseteq (k, w), (\tau_1, \tau_2, r) \in T^{\neg\iota}_{k'}[\![\Omega]\!]w'. \\
& & \quad (k'', w'', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E^\iota_n[\![\tau]\!]\rho, \alpha \mapsto r\} \\
V^\iota_n[\![\exists \alpha.\tau]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, \mathsf{pack}\langle \tau_1, v_1 \rangle, \mathsf{pack}\langle \tau_2, v_2 \rangle) \in \text{Atom}_n[\rho^1(\exists\alpha.\tau), \rho^2(\exists\alpha.\tau)] \mid \\
& & \quad \exists r.\ (\tau_1, \tau_2, r) \in T^\iota_k[\![\Omega]\!]w \wedge \forall (k', w') \sqsupset (k, w). \\
& & \quad (k', w', v_1, v_2) \in V^\iota_n[\![\tau]\!]\rho, \alpha \mapsto r\} \\
E^\iota_n[\![\tau]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, e_1, e_2) \in \text{Atom}_n[\rho^1(\tau), \rho^2(\tau)] \mid \\
& & \quad \forall j < k, v_1, \sigma_1.\ w.\sigma_1; e_1 \hookrightarrow^j \sigma_1; v_1 \implies \\
& & \quad \exists v_2, w'.\ (k - j, w') \sqsupseteq (k, w) \wedge w'.\sigma_1 = \sigma_1 \wedge \\
& & \quad w.\sigma_2; e_2 \hookrightarrow^* w'.\sigma_2; v_2 \wedge (k - j, w', v_1, v_2) \in V^\iota_n[\![\tau]\!]\rho\} \\
T^{\bullet,-}_n[\![\Omega]\!]w & \stackrel{\text{def}}{=} & \{(w.\eta^1(\tau), w.\eta^2(\tau), (w.\rho^1(\tau), w.\rho^2(\tau), V^\bullet_n[\![\tau]\!]w.\rho)) \mid \\
& & \quad \tau \in \text{Typ}_{\text{dom}(w.\rho)}\} \\
T^{\circ,+}_n[\![\Omega]\!]w & \stackrel{\text{def}}{=} & \{(\tau_1, \tau_2, (w.\sigma_1^*(\tau_1), w.\sigma_2^*(\tau_2), R)) \mid \\
& & \quad \tau_1 \in \text{Typ}_{\text{dom}(w.\sigma_1)} \wedge \tau_2 \in \text{Typ}_{\text{dom}(w.\sigma_2)} \wedge R \in \text{Rel}_n[w.\sigma_1^*(\tau_1), w.\sigma_2^*(\tau_2)]\} \\
D^\iota_n[\![\epsilon]\!]w & \stackrel{\text{def}}{=} & \{(\emptyset, \emptyset, \emptyset)\} \\
D^\iota_n[\![\Delta, \alpha]\!]w & \stackrel{\text{def}}{=} & \{((\delta_1, \alpha \mapsto \tau_1), (\delta_2, \alpha \mapsto \tau_2), (\rho, \alpha \mapsto r)) \mid \\
& & \quad (\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w \wedge (\tau_1, \tau_2, r) \in T^\iota_n[\![\Omega]\!]w\} \\
D^\iota_n[\![\Delta, \alpha {\approx} \tau]\!]w & \stackrel{\text{def}}{=} & \{((\delta_1, \alpha \mapsto \alpha_1), (\delta_2, \alpha \mapsto \alpha_2), (\rho, \alpha \mapsto (\rho^1(\tau), \rho^2(\tau), V^{|\iota|}_n[\![\tau]\!]\rho))) \mid \\
& & \quad (\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w \wedge w.\sigma_i(\alpha_i) = \delta_i(\tau) \wedge \\
& & \quad \exists \alpha'.\ \alpha_i = w.\eta^i(\alpha') \wedge w.\rho(\alpha').R = V^{|\iota|}_n[\![\tau]\!]\rho\} \\
G^\iota_n[\![\epsilon]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, \emptyset, \emptyset) \mid k < n \wedge w \in \text{World}_k\} \\
G^\iota_n[\![\Gamma, x{:}\tau]\!]\rho & \stackrel{\text{def}}{=} & \{(k, w, (\gamma_1, x \mapsto v_1), (\gamma_2, x \mapsto v_2)) \mid \\
& & \quad (k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho \wedge (k, w, v_1, v_2) \in V^\iota_n[\![\tau]\!]\rho\}
\end{array}
$$

$$
\begin{array}{l}
\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau \iff^{\text{def}} \Delta; \Gamma \vdash e_1 : \tau \wedge \Delta; \Gamma \vdash e_2 : \tau \wedge \\
\quad \forall n \geq 0, w_0 \in \text{World}_n, (\delta_1, \delta_2, \rho) \in D^{\neg\iota}_n[\![\Delta]\!]w_0, (k, w, \gamma_1, \gamma_2) \in G^{\neg\iota}_n[\![\Gamma]\!]\rho. \\
\quad (k, w) \sqsupset (n, w_0) \implies (k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E^\iota_n[\![\tau]\!]\rho
\end{array}
$$

2.3 Properties

Note: whenever we write $V_n^\iota[\![\tau]\!]\rho$ or $E_n^\iota[\![\tau]\!]\rho$ from now on, we assume that $\tau \in \mathrm{dom}(\rho)$ and $\rho \in \mathrm{Interp}$.

Lemma 2.1 (Transitivity of World Extension)

1. If $(k'', w'') \sqsupseteq (k', w')$ and $(k', w') \sqsupseteq (k, w)$, then $(k'', w'') \sqsupseteq (k, w)$.
2. If $(k'', w'') \sqsupset (k', w')$ and $(k', w') \sqsupset (k, w)$, then $(k'', w'') \sqsupset (k, w)$.

Proof:

1. $k'' \le k$ is clear, as are $w'' \in \mathrm{World}_{k''}$, $w''.\sigma_i \sqsupseteq w.\sigma_i$, $w''.\eta \sqsupseteq w.\eta$, and $\mathrm{rng}(w''.\eta^i) \setminus \mathrm{rng}(w.\eta^i) \subseteq \mathrm{dom}(w''.\sigma_i) \setminus \mathrm{dom}(w.\sigma_i)$. It remains to show $\rho'' \sqsupseteq \lfloor\rho\rfloor_{k''}$. So suppose $\alpha \in \mathrm{dom}(\lfloor\rho\rfloor_{k''})$.

$$\begin{array}{ll}
\lfloor\rho\rfloor_{k''}(\alpha).R & \\
= \lfloor\rho(\alpha).R\rfloor_{k''} & \\
= \lfloor\lfloor\rho(\alpha).R\rfloor_{k'}\rfloor_{k''} & \text{since } k'' \le k' \\
= \lfloor\lfloor\rho\rfloor_{k'}(\alpha).R\rfloor_{k''} & \\
= \lfloor\rho'(\alpha).R\rfloor_{k''} & \text{since } \rho' \sqsupseteq \lfloor\rho\rfloor_{k'} \\
= \lfloor\rho'\rfloor_{k''}(\alpha).R & \\
= \rho''(\alpha).R & \text{since } \rho'' \sqsupseteq \lfloor\rho'\rfloor_{k''}
\end{array}$$

2. Follows from (1).

■

Lemma 2.2
If $(\delta_1, \delta_2, \rho) \in D_n^\iota[\![\Delta]\!]w$, then $\rho^i = w.\sigma_i^* \cdot \delta_i$.

Lemma 2.3 (Inclusion)
$V_n^\iota[\![\tau]\!]\rho \subseteq E_n^\iota[\![\tau]\!]\rho$

Proof: Follows easily from the definition of $E_n^\iota[\![\tau]\!]\rho$, taking the future world to be the current one.
■

Lemma 2.4 (Restriction)

1. If $k' \le k$, then $V_{k'}^\iota[\![\tau]\!]\rho = \lfloor V_k^\iota[\![\tau]\!]\rho\rfloor_{k'}$.
2. If $k' \le k$, then $E_{k'}^\iota[\![\tau]\!]\rho = \lfloor E_k^\iota[\![\tau]\!]\rho\rfloor_{k'}$.

Lemma 2.5 (Atomicity)

1. $V_n^\iota[\![\tau]\!]\rho \subseteq \mathrm{Atom}_n^{\mathrm{val}}[\rho^1(\tau), \rho^2(\tau)]$.
2. $E_n^\iota[\![\tau]\!]\rho \subseteq \mathrm{Atom}_n[\rho^1(\tau), \rho^2(\tau)]$.

Proof: By definition. ∎

Lemma 2.6 (Irrelevance)
If $\lfloor\rho'\rfloor_n \sqsupseteq \lfloor\rho\rfloor_n$, then

1. $V_n^\iota[\![\tau]\!]\rho' = V_n^\iota[\![\tau]\!]\rho$,
2. $E_n^\iota[\![\tau]\!]\rho' = E_n^\iota[\![\tau]\!]\rho$, and
3. $G_n^\iota[\![\tau]\!]\rho' = G_n^\iota[\![\tau]\!]\rho$.

Proof: (1) and (2) by mutual induction on τ.

1. - Case $\tau = \alpha$:

$$\begin{array}{ll} (k, w, v_1, v_2) \in V_n^\iota[\![\alpha]\!]\rho' & \\ \iff (k, w, v_1, v_2) \in \lfloor\rho'(\alpha).R\rfloor_n & \\ \iff (k, w, v_1, v_2) \in \lfloor\rho'\rfloor_n(\alpha).R & \\ \iff (k, w, v_1, v_2) \in \lfloor\rho\rfloor_n(\alpha).R & \text{by assumption} \\ \iff (k, w, v_1, v_2) \in V_n^\iota[\![\alpha]\!]\rho & \end{array}$$

   - Case $\tau = b$: Trivial.
   - The remaining cases all follow easily by induction.
2. Follows from (1).
3. Follows from (1).

∎

Lemma 2.7

1. If $(\tau_1, \tau_2, r) \in T_n^\iota[\![\Omega]\!]w_0$ and $(k, w) \sqsupseteq (n, w_0)$, then $(\tau_1, \tau_2, \lfloor r\rfloor_k) \in T_k^\iota[\![\Omega]\!]w$.
2. If $(\delta_1, \delta_2, \rho) \in D_n^\iota[\![\Delta]\!]w_0$ and $(k, w) \sqsupseteq (n, w_0)$, then $(\delta_1, \delta_2, \lfloor\rho\rfloor_k) \in D_k^\iota[\![\Delta]\!]w$.

Proof:

1. Follows easily by Restriction (2.4) and Irrelevance (2.6).
2. By induction on Δ.
   - Case $\Delta = \epsilon$: Trivial.
   - Case $\Delta = \Delta', \alpha$: Follows by induction and part (1).
   - Case $\Delta = \Delta', \alpha{\approx}\tau$:
     - We know $\delta_i = \delta_i', \alpha{\mapsto}\alpha_i$ and $\rho = \rho', \alpha{\mapsto}(\rho^1(\tau), \rho^2(\tau), V_n^{|\iota|}[\![\tau]\!]\rho')$ where $(\delta_1', \delta_2', \rho') \in D_n^\iota[\![\Delta']\!]w_0$, $w_0.\sigma_i(\alpha_i) = \delta_i(\tau)$, and $w_0.\rho(\alpha').R = V_n^{|\iota|}[\![\tau]\!]\rho'$ for some α'.

– By induction, $(\delta_1', \delta_2', \lfloor\rho'\rfloor) \in D_k^\iota[\![\Delta']\!]w$.
– $w.\sigma_i(\alpha_i) = \delta_i(\tau)$ is clear.
– We need to show $w.\rho(\alpha').R = \lfloor V_n^{|\iota|}[\![\tau]\!]\lfloor\rho'\rfloor_k\rfloor_k$ and $\lfloor V_n^{|\iota|}[\![\tau]\!]\rho'\rfloor_k = V_k^{|\iota|}[\![\tau]\!]\lfloor\rho'\rfloor_k$.
– Both follow by Restriction (2.4) and Irrelevance (2.6), the former using the fact that $w \in \text{World}_k$.

■

Lemma 2.8 (Closure Under World Extension)

1. If $(k, w, v_1, v_2) \in V_n^\iota[\![\tau]\!]\rho$ and $(k', w') \sqsupseteq (k, w)$, then $(k', w', v_1, v_2) \in V_n^\iota[\![\tau]\!]\rho$.
2. If $(k, w, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$ and $(k', w') \sqsupseteq (k, w)$, then $(k', w', \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$.

Proof:

1. By induction on τ.
   - Case $\tau = \alpha$: Since $\rho \in \text{Interp}$, we know $\rho(\alpha).R \in \text{Rel}_m$ for some m and thus closed under world extension.
   - Case $\tau = b$: Trivial.
   - Case $\tau = \tau_1 \times \tau_2$: Follows easily by induction.
   - Case $\tau = \tau_1 \to \tau_2$: $v_i = \lambda x{:}\tau_i'.e_i$.
     – So suppose $(k'', w'', v_3, v_4) \in V_n^{\neg\iota}[\![\tau_1]\!]\rho$ where $(k'', w'') \sqsupseteq (k', w')$.
     – To show: $(k'', w'', e_1[v_3/x], e_2[v_4/x]) \in E_n^\iota[\![\tau_2]\!]\rho$.
     – By Transitivity of World Extension (2.1), $(k'', w'') \sqsupseteq (k, w)$.
     – Now instantiate the assumption to get $(k'', w'', e_1[v_3/x], e_2[v_4/x]) \in E_n^\iota[\![\tau_2]\!]\rho$.
   - Case $\tau = \forall\alpha.\tau'$: $v_i = \Lambda\alpha.e_i$.
     – So suppose $(k''', w''') \sqsupset (k'', w'') \sqsupseteq (k', w')$ and $(\tau_1, \tau_2, r) \in T_{k''}^{\neg\iota}[\![\Omega]\!]w''$.
     – To show: $(k''', w''', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^\iota[\![\tau']\!]\rho, \alpha{\mapsto}r$.
     – By Transitivity of World Extension (2.1), $(k'', w'') \sqsupseteq (k, w)$.
     – Now instantiating the assumption yields the claim.
   - Case $\tau = \exists\alpha.\tau'$: $v_i = \mathsf{pack}\,\langle\tau_i, v_i'\rangle$.
     – By assumption there is r such that $(\tau_1, \tau_2, r) \in T_k^\iota[\![\Omega]\!]w$ and $(k'', w'', v_1', v_2') \in V_n^\iota[\![\tau']\!]\rho, \alpha{\mapsto}r$ for any $(k'', w'') \sqsupset (k, w)$.
     – By Lemma 2.7 we have $(\tau_1, \tau_2, \lfloor r\rfloor_{k'}) \in T_{k'}^\iota[\![\Omega]\!]w'$.
     – We show that $(k'', w'', v_1', v_2') \in V_n^\iota[\![\tau']\!]\rho, \alpha{\mapsto}\lfloor r\rfloor_{k'}$ for any $(k'', w'') \sqsupset (k', w')$.
     – Suppose $(k'', w'') \sqsupset (k', w')$.
     – By Transitivity of World Extension (2.1), $(k'', w'') \sqsupset (k, w)$.
     – Hence $(k'', w'', v_1', v_2') \in V_n^\iota[\![\tau']\!]\rho, \alpha{\mapsto}r$ and thus the claim follows by Restriction (2.4) and Irrelevance (2.6).
2. Follows from (1).

■

Lemma 2.9 (Validity of the Logical Relation)
$V^\iota_n[\![\tau]\!]\rho \in \mathrm{Rel}_n[\rho^1(\tau), \rho^2(\tau)]$

Proof: Follows by Atomicity (2.5) and Closure Under World Extension (2.8). ■

Lemma 2.10 (Substitution)
For $\iota \in \{\bullet, \circ\}$:

1. $V^\iota_n[\![\tau]\!]\rho, \alpha\mapsto(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho) = V^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$.
2. $E^\iota_n[\![\tau]\!]\rho, \alpha\mapsto(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho) = E^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$.

Proof:

1. - Case $\tau = \alpha$: Trivial.
   - Case $\tau = \alpha' \neq \alpha$: Follows by Irrelevance (2.6).
   - Case $\tau = \tau_0 \times \tau_0'$: Follows easily by induction.
   - Case $\tau = \tau_0' \to \tau_0$: Follows easily by induction.
   - Case $\tau = \forall\alpha'.\tau''$:
     - We show $V^\iota_n[\![\tau]\!]\rho, \alpha\mapsto(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho) \subseteq V^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$. The other direction is symmetric.
     - Suppose $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V^\iota_n[\![\tau]\!]\rho, \alpha\mapsto(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho)$.
     - Suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T^{\neg\iota}_{k'}[\![\Omega]\!]w'$.
     - To show: $(k'', w'', e_1[\tau_1/\alpha'], e_2[\tau_2/\alpha']) \in E^\iota_n[\![\tau''[\tau'/\alpha]]\!]\rho, \alpha'\mapsto r$
     - Since $\neg\iota = \iota$ and thus $T^{\neg\iota}_{k'}[\![\Omega]\!]w' = T^\iota_{k'}[\![\Omega]\!]w'$, we know: $(k'', w'', e_1[\tau_1/\alpha'], e_2[\tau_2/\alpha']) \in E^\iota_n[\![\tau'']\!]\rho, \alpha\mapsto(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho), \alpha'\mapsto r$
     - By Irrelevance (2.6),
       $(\rho^1(\tau'), \rho^2(\tau'), V^\iota_n[\![\tau']\!]\rho) = ((\rho, \alpha\mapsto r)^1(\tau'), (\rho, \alpha\mapsto r)^2(\tau'), V^\iota_n[\![\tau']\!]\rho, \alpha\mapsto r)$.
     - Hence by induction, $(k'', w'', e_1[\tau_1/\alpha'], e_2[\tau_2/\alpha']) \in E^\iota_n[\![\tau''[\tau'/\alpha]]\!]\rho, \alpha'\mapsto r$.
   - Case $\tau = \exists\alpha'.\tau''$: Analogously to the previous case.
2. Follows from part (1).

■

Lemma 2.11 (Type Compatibility)
For $\iota \in \{\bullet, \circ\}$: If $\Delta \vdash \tau_1 \approx \tau_2$ and $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w$, then

1. $V^\iota_n[\![\tau_1]\!]\rho = V^\iota_n[\![\tau_2]\!]\rho$ and
2. $E^\iota_n[\![\tau_1]\!]\rho = E^\iota_n[\![\tau_2]\!]\rho$.

Proof: By induction on the derivation of the type compatibility.

1. - Case $\tau_1 = \alpha = \tau_2$: Trivial.
   - Case $\tau_1 = \alpha$ where $\alpha{\approx}\tau_2 \in \Delta$:
     - Then it is easy to see that $\Delta = \Delta_1, \alpha{\approx}\tau_2, \Delta_2$, $\delta_i = \delta_{i1}, \alpha{\mapsto}\alpha_i, \delta_{i2}$, and $\rho = \rho_1, \alpha{\mapsto}(\rho_1^1(\tau_2), \rho_1^2(\tau_2), V_n^\iota[\![\tau_2]\!]\rho_1), \rho_2$.
     - Hence $V_n^\iota[\![\alpha]\!]\rho = V_n^\iota[\![\tau_2]\!]\rho_1$ and so the claim follows by Irrelevance (2.6).
   - Case $\tau_1 = b = \tau_2$: Trivial.
   - Case $\tau_1 = \tau_{11} \times \tau_{12}$ and $\tau_2 = \tau_{21} \times \tau_{22}$ with $\Delta \vdash \tau_{11} \approx \tau_{21}$ and $\Delta \vdash \tau_{21} \approx \tau_{22}$: Follows easily by induction.
   - Case $\tau_1 = \tau_{11} \to \tau_{12}$ and $\tau_2 = \tau_{21} \to \tau_{22}$ with $\Delta \vdash \tau_{11} \approx \tau_{21}$ and $\Delta \vdash \tau_{12} \approx \tau_{22}$: Follows easily by induction.
   - Case $\tau_1 = \forall\alpha.\tau_1'$ and $\tau_2 = \forall\alpha.\tau_2'$ with $\Delta, \alpha \vdash \tau_1' \approx \tau_2'$:
     - We show $V_n^\iota[\![\delta(\tau_1)]\!]\rho \subseteq V_n^\iota[\![\delta(\tau_2)]\!]\rho$; the other direction is symmetric.
     - Suppose $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^\iota[\![\forall\alpha.\tau_1']\!]\rho$.
     - Suppose further $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T_{k'}^{\neg\iota}[\![\Omega]\!]w'$.
     - To show: $(k'', w'', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^\iota[\![\tau_2']\!]\rho, \alpha{\mapsto}r$.
     - Since $\neg\iota = \iota$ and thus $T_{k'}^{\neg\iota}[\![\Omega]\!]w' = T_{k'}^{\iota}[\![\Omega]\!]w'$, we know $(k'', w'', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^\iota[\![\tau_1']\!]\rho, \alpha{\mapsto}r$.
     - By Restriction (2.4) and Irrelevance (2.6), this reduces to showing $E_{k'}^\iota[\![\tau_1']\!]\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r = E_{k'}^\iota[\![\tau_2']\!]\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r$.
     - This follows by induction if we can show $((\delta_1, \alpha{\mapsto}\tau_1), (\delta_2, \alpha{\mapsto}\tau_2), (\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r)) \in D_{k'}^\iota[\![\Delta, \alpha]\!]w'$.
     - By Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k'}) \in D_{k'}^\iota[\![\Delta]\!]w'$.
     - Hence indeed $((\delta_1, \alpha{\mapsto}\tau_1), (\delta_2, \alpha{\mapsto}\tau_2), (\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r)) \in D_{k'}^\iota[\![\Delta, \alpha]\!]w'$.
   - Case $\tau_1 = \exists\alpha.\tau_1'$ and $\tau_2 = \exists\alpha.\tau_2'$ with $\Delta, \alpha \vdash \tau_1' \approx \tau_2'$: analogously to the previous case
2. Follows immediately from part (1).

■

Definition 2.12 (Anti-Unifier)
Assume that $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\Delta]\!]w$. The anti-unifying substitution of δ_1 and δ_2 with respect to $w.\eta$, written $\mathrm{au}(\delta_1, \delta_2, w.\eta)$, is defined as follows.

$$\mathrm{au}(\epsilon, \epsilon, \eta) \stackrel{\text{def}}{=} \epsilon$$
$$\mathrm{au}((\delta_1, \alpha{\mapsto}\tau_1), (\delta_2, \alpha{\mapsto}\tau_2), \eta) \stackrel{\text{def}}{=} \mathrm{au}(\delta_1, \delta_2, \eta), \alpha{\mapsto}\tau \qquad \text{where } \tau = \eta^{-1}(\tau_1) = \eta^{-2}(\tau_2)$$

Note that η^{-i} exists because the definition of Conc ensures that η^i is injective.

Lemma 2.13

1. If $\delta = \mathrm{au}(\delta_1, \delta_2, \eta)$, then $\delta_1 = \eta^1 \cdot \delta$ and $\delta_2 = \eta^2 \cdot \delta$.

2. If $(\delta_1, \delta_2, \rho) \in D^\bullet_n[\![\Delta]\!]w_0$, $\delta = \mathrm{au}(\delta_1, \delta_2, w_0.\eta)$, and $(k, w) \sqsupseteq (n, w_0)$, then $\delta = \mathrm{au}(\delta_1, \delta_2, w.\eta)$.

Proof:

1. Follows easily from the definition.

■

Lemma 2.14
If $(\delta_1, \delta_2, \rho) \in D^\bullet_n[\![\Delta]\!]w_0$, $\delta = \mathrm{au}(\delta_1, \delta_2, w_0.\eta)$ and $\Delta \vdash \tau$, then

1. $V^\bullet_n[\![\tau]\!]\rho = V^\bullet_n[\![\delta(\tau)]\!]w_0.\rho$ and
2. $E^\bullet_n[\![\tau]\!]\rho = E^\bullet_n[\![\delta(\tau)]\!]w_0.\rho$.

Proof: By induction on the derivation of $\Delta \vdash \tau$.

1.
   - Case $\tau = \alpha$ were $\alpha \in \Delta$:
     - Then it is easy to see that $\Delta = \Delta_1, \alpha, \Delta_2$, $\delta_i = \delta_{i1}, \alpha{\mapsto}\tau_i, \delta_{i2}$, and $\rho = \rho_1, \alpha{\mapsto}r, \rho_2$ where $(\tau_1, \tau_2, r) \in T^\bullet_n[\![\Omega]\!]w_0$.
     - The latter implies $\tau_i = w_0.\eta^i(\tau')$ and $r = (\_, \_, V^\bullet_n[\![\tau']\!]w_0.\rho)$.
     - Hence $V^\bullet_n[\![\alpha]\!]\rho = V^\bullet_n[\![\tau']\!]w_0.\rho = V^\bullet_n[\![\delta(\alpha)]\!]w_0.\rho$.
   - Case $\tau = \alpha$ where $\alpha{\approx}\tau' \in \Delta$:
     - Then it is easy to see that $\Delta = \Delta_1, \alpha{\approx}\tau', \Delta_2$, $\delta_i = \delta_{i1}, \alpha{\mapsto}\alpha_i, \delta_{i2}$, and $\rho = \rho_1, \alpha{\mapsto}(\rho^1_1(\tau'), \rho^2_1(\tau'), V^\bullet_n[\![\tau']\!]\rho_1), \rho_2$ with $\alpha_i = w_0.\eta^i(\alpha')$ and $w_0.\rho(\alpha').R = V^\bullet_n[\![\tau']\!]\rho_1$ for some α'.
     - Because of the injectivity of $w_0.\eta^i$, $w_0.\eta^i(\alpha') = \alpha_i = \delta_i(\alpha) = w_0.\eta^i\delta(\alpha)$ implies $\alpha' = \delta(\alpha)$.
     - Hence $V^\bullet_n[\![\alpha]\!]\rho = V^\bullet_n[\![\tau']\!]\rho_1 = V^\bullet_n[\![\alpha']\!]w_0.\rho = V^\bullet_n[\![\delta(\alpha)]\!]w_0.\rho$.
   - Case $\tau = b$: Trivial.
   - Case $\tau = \tau_1 \times \tau_2$ with $\Delta \vdash \tau_1$ and $\Delta \vdash \tau_2$: Follows easily by induction.
   - Case $\tau = \tau_1 \to \tau_2$ with $\Delta \vdash \tau_1$ and $\Delta \vdash \tau_2$: Follows easily by induction.
   - Case $\tau = \forall\alpha.\tau'$ with $\Delta, \alpha \vdash \tau'$:
     - We show $V^\bullet_n[\![\tau]\!]\rho \subseteq V^\bullet_n[\![\delta(\tau)]\!]w_0.\rho$; the other direction is symmetric.
     - Suppose $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V^\bullet_n[\![\forall\alpha.\tau']\!]\rho$.
     - Suppose further $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T^\bullet_{k'}[\![\Omega]\!]w'$.
     - We know $(k'', w'', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E^\bullet_n[\![\tau']\!]\rho, \alpha{\mapsto}r$.
     - To show: $(k'', w'', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E^\bullet_n[\![\delta(\tau')]\!]w_0.\rho, \alpha{\mapsto}r$
     - By Restriction (2.4) and Irrelevance (2.6), this reduces to showing $E^\bullet_{k'}[\![\tau']\!]\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r = E^\bullet_{k'}[\![\delta(\tau')]\!]w'.\rho, \alpha{\mapsto}r$.
     - By assumption and Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k'}) \in D^\bullet_{k'}[\![\Delta]\!]w'$.
     - Let $(\delta'_1, \delta'_2, \rho') := ((\delta_1, \alpha{\mapsto}\tau_1), (\delta_2, \alpha{\mapsto}\tau_2), (\lfloor\rho\rfloor_{k'}, \alpha{\mapsto}r))$, so $(\delta'_1, \delta'_2, \rho') \in D^\bullet_{k'}[\![\Delta, \alpha]\!]w'$.
     - By Lemma 2.13, $\delta = \mathrm{au}(\delta_1, \delta_2, w'.\eta)$.

- Since $(\tau_1, \tau_2, r) \in T^{\bullet}_{k'}[\![\Omega]\!]w'$ we know $\tau_i = w'.\eta^i(\tau'')$ and $r = (w'.\rho^1(\tau''), w'.\rho^2(\tau''), V^{\bullet}_{k'}[\![\tau'']\!]w'.\rho)$.
- It is easy to see then that $\delta, \alpha{\mapsto}\tau'' = \mathrm{au}(\delta'_1, \delta'_2, w'.\eta)$.
- Hence by induction, $E^{\bullet}_{k'}[\![\tau']\!]\rho' = E^{\bullet}_{k'}[\![\delta(\tau')[\tau''/\alpha]]\!]w'.\rho$.
- By Substitution (2.10), $E^{\bullet}_{k'}[\![\delta(\tau')[\tau''/\alpha]]\!]w'.\rho =$ $E^{\bullet}_{k'}[\![\delta(\tau')]\!]w'.\rho, \alpha{\mapsto}(w'.\rho^1(\tau''), w'.\rho^2(\tau''), V^{\bullet}_{k'}[\![\tau'']\!]w'.\rho) = E^{\bullet}_{k'}[\![\delta(\tau')]\!]w'.\rho, \alpha{\mapsto}r$.

- Case $\tau = \exists\alpha.\tau'$ with $\Delta, \alpha \vdash \tau'$: analogously to previous case

2. Follows immediately from part (1).

■

In all the following compatibility lemmas we assume $\iota \in \{\bullet, \circ\}$.

Lemma 2.15 (Compatibility: var)
If $\Delta \vdash \Gamma$ and $x{:}\tau \in \Gamma$, then $\Delta; \Gamma \vdash x \precsim^{\iota} x : \tau$.

Proof:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D^{\iota}_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^{\iota}_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupseteq (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(x), \delta_2\gamma_2(x)) \in E^{\iota}_n[\![\tau]\!]\rho$
- By Inclusion (2.3) it suffices to show $(k, w, \delta_1\gamma_1(x), \delta_2\gamma_2(x)) \in V^{\iota}_n[\![\tau]\!]\rho$.
- By assumption we have $(k, w, \gamma_1(x), \gamma_2(x)) \in V^{\iota}_n[\![\tau]\!]\rho$.
- Since thereby $\vdash w.\sigma_i; \gamma_i(x) : \rho^1(\tau)$, we know $\gamma_i(x) = \delta_i\gamma_i(x)$.

■

Lemma 2.16 (Compatibility: con)
If $\Delta \vdash \Gamma$, then $\Delta; \Gamma \vdash c \precsim^{\iota} c : b$.

Proof:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D^{\iota}_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^{\iota}_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupseteq (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(c), \delta_2\gamma_2(c) \in E^{\iota}_n[\![b]\!]\rho$, i.e., $(k, w, c, c) \in E^{\iota}_n[\![b]\!]\rho$.
- By Inclusion (2.3) it suffices to show $(k, w, c, c) \in V^{\iota}_n[\![b]\!]\rho$, which holds trivially.

■

Lemma 2.17 (Compatibility: pair)
If $\Delta; \Gamma \vdash e_1 \precsim^{\iota} e_2 : \tau_1$ and $\Delta; \Gamma \vdash e_3 \precsim^{\iota} e_4 : \tau_2$, then $\Delta; \Gamma \vdash \langle e_1, e_3\rangle \precsim^{\iota} \langle e_2, e_4\rangle : \tau_1 \times \tau_2$.

Proof:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D^{\iota}_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^{\iota}_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupseteq (n, w_0)$.

- To show: $(k, w, \delta_1\gamma_1(\langle e_1, e_3\rangle), \delta_2\gamma_2(\langle e_2, e_4\rangle)) \in E_n^\iota[\![\tau_1 \times \tau_2]\!]\rho$.
- Assume that $w.\sigma_1; \delta_1\gamma_1(\langle e_1, e_3\rangle)$ terminates in $j_1 + j_2 =: j < k$ steps:

$$\begin{array}{ll} & w.\sigma_1; \delta_1\gamma_1(\langle e_1, e_3\rangle) \\ \hookrightarrow^{j_1} & \sigma_1'; \langle v_1, \delta_1(\gamma_1(e_3))\rangle \\ \hookrightarrow^{j_2} & \sigma_1; \langle v_1, v_3\rangle \end{array}$$

- Instantiate the first assumption to get $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E_n^\iota[\![\tau_1]\!]\rho$.
- Consequently there exists $(k - j_1, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(\langle e_2, e_4\rangle) \hookrightarrow^* w'.\sigma_2; \langle v_2, \delta_2\gamma_2(e_4)\rangle$ with $w'.\sigma_1 = \sigma_1'$ and $(k - j_1, w', v_1, v_2) \in V_n^\iota[\![\tau_1]\!]\rho$.
- Instantiate the second assumption and apply Closure Under World Extension (2.8) to get $(k - j_1, w', \delta_1\gamma_1(e_3), \delta_2\gamma_2(e_4)) \in E_n^\iota[\![\tau_2]\!]\rho$.
- Consequently there exists $(k - j, w'') \sqsupseteq (k - j_1, w')$ such that $w'.\sigma_2; \langle v_2, \delta_2\gamma_2(e_4)\rangle \hookrightarrow^* w''.\sigma_2; \langle v_2, v_4\rangle$ with $w''.\sigma_1 = \sigma_1$ and $(k - j, w'', v_3, v_4) \in V_n^\iota[\![\tau_2]\!]\rho$.
- Since, by Closure Under World Extension, also $(k - j, w'', v_1, v_2) \in V_n^\iota[\![\tau_1]\!]\rho$, we get $(k - j, w'', \langle v_1, v_3\rangle, \langle v_2, v_4\rangle) \in V_n^\iota[\![\tau_1 \times \tau_2]\!]\rho$.

■

Lemma 2.18 (Compatibility: proj)
If $\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau_1 \times \tau_2$, then $\Delta; \Gamma \vdash e_1.i \precsim^\iota e_2.i : \tau_i$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\iota[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(e_1.i), \delta_2\gamma_2(e_2.i)) \in E_n^\iota[\![\tau_i]\!]\rho$.
- Assume that $w.\sigma_1; \delta_1\gamma_1(e_1.i)$ terminates in $j' + 1 =: j < k$ steps:

$$\begin{array}{ll} & w.\sigma_1; \delta_1\gamma_1(e_1.i) \\ \hookrightarrow^{j'} & \sigma_1; \langle v_{11}, v_{12}\rangle.i \\ \hookrightarrow^{1} & \sigma_1; v_{1i} \end{array}$$

- Instantiate the assumption to get $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E_n^\iota[\![\tau_1 \times \tau_2]\!]\rho$.
- Consequently there exists $(k - j', w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(e_2.i) \hookrightarrow^* w'.\sigma_2; \langle v_{21}, v_{22}\rangle.i$ with $w'.\sigma_1 = \sigma_1$ and $(k - j', w', \langle v_{11}, v_{12}\rangle, \langle v_{21}, v_{22}\rangle) \in V_n^\iota[\![\tau_1 \times \tau_2]\!]\rho$.
- By Closure Under World Extension (2.8), $(k - j, \lfloor w'\rfloor_{k-j}, \langle v_{11}, v_{12}\rangle, \langle v_{21}, v_{22}\rangle) \in V_n^\iota[\![\tau_1 \times \tau_2]\!]\rho$.
- Hence $(k - j, \lfloor w'\rfloor_{k-j}, v_{1i}, v_{2i}) \in V_n^\iota[\![\tau_i]\!]\rho$.

■

Lemma 2.19 (Compatibility: abs)
If $\Delta; \Gamma, x{:}\tau_1 \vdash e_1 \precsim^\iota e_2 : \tau_2$, then $\Delta; \Gamma \vdash \lambda x{:}\tau_1.e_1 \precsim^\iota \lambda x{:}\tau_1.e_2 : \tau_1 \to \tau_2$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1(\gamma_1(\lambda x{:}\tau_1.e_1)), \delta_2(\gamma_2(\lambda x{:}\tau_1.e_2))) \in E^\iota_n[\![\tau_1 \to \tau_2]\!]\rho$
- By Inclusion (2.3) it suffices to show $(k, w, \delta_1(\gamma_1(\lambda x{:}\tau_1.e_1)), \delta_2(\gamma_2(\lambda x{:}\tau_1.e_2))) \in V^\iota_n[\![\tau_1 \to \tau_2]\!]\rho$.
- So suppose $(k', w', v_1, v_2) \in V^\iota_n[\![\tau_1]\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.
- To show: $(k', w', \delta_1(\gamma_1(e_1))[v_1/x], \delta_2(\gamma(e_2))[v_2/x]) \in E^\iota_n[\![\tau_2]\!]\rho$
- Let $\gamma'_i := \gamma_i, x{\mapsto}v_i$.
- As $(k', w', \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ by Closure Under World Extension (2.8), this means $(k', w', \gamma'_1, \gamma'_2) \in G^\iota_n[\![\Gamma, x{:}\tau_1]\!]\rho$.
- Now instantiate the assumption to get $(k', w', \delta_1(\gamma'_1(e_1)), \delta_2(\gamma'_2(e_2))) \in E^\iota_n[\![\tau_2]\!]\rho$.
- Note that $\delta_i(\gamma'_i(e_i)) = \delta_i(\gamma_i(e_i)[v_i/x])$.
- Furthermore, since $(k', w', v_1, v_2) \in V^\iota_n[\![\tau_1]\!]\rho$ implies $\vdash w'.\sigma_i; v_i : \rho^i(\tau_1)$, we have $\delta_i(v_i) = v_i$ and thus $\delta_i(\gamma_i(e_i)[v_i/x]) = \delta_i(\gamma_i(e_i))[v_i/x]$.

■

Lemma 2.20 (Compatibility: app)
If $\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau_1 \to \tau_2$ and $\Delta; \Gamma \vdash e_3 \precsim^\iota e_4 : \tau_1$, then $\Delta; \Gamma \vdash e_1\, e_3 \precsim^\iota e_2\, e_4 : \tau_2$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(e_1\, e_3), \delta_2\gamma_2(e_2\, e_4)) \in E^\iota_n[\![\tau_2]\!]\rho$
- Assume that $w.\sigma_1; \delta_1\gamma_1(e_1\, e_3)$ terminates in $j_1 + j_2 + 1 + j_3 =: j < k$ steps:

$$\begin{array}{rl} & w.\sigma_1; \delta_1\gamma_1(e_1\, e_3) \\ \hookrightarrow^{j_1} & \sigma'_1; (\lambda x.e'_1)\, \delta_1(\gamma_1(e_3)) \\ \hookrightarrow^{j_2} & \sigma''_1; (\lambda x.e'_1)\, v_3 \\ \hookrightarrow^{1} & \sigma''_1; e'_1[v_3/x] \\ \hookrightarrow^{j_3} & \sigma_1; v_1 \end{array}$$

- Instantiating the first assumption yields the existence of $(k - j_1, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(e_2\, e_4) \hookrightarrow^* w'.\sigma_2; (\lambda x.e'_2)\, \delta_2\gamma_2(e_4)$ and $w'.\sigma_1 = \sigma'_1$ and $(k - j_1, w', \lambda x.e'_1, \lambda x.e'_2) \in V^\iota_n[\![\tau_1 \to \tau_2]\!]\rho$.
- By Closure Under World Extension (2.8) we have $(k - j_1, w', \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$.

- Hence instantiating the second assumption yields the existence of $(k - j_1 - j_2, w'') \sqsupseteq (k - j_1, w')$ such that $w'.\sigma_2; (\lambda x.e'_2)\, \delta_2(\gamma_2(e_4)) \hookrightarrow^* w''.\sigma_2; (\lambda x.e'_2)\, v_4$ and $w''.\sigma_1 = \sigma''_1$ and $(k - j_1 - j_2, w'', v_3, v_4) \in V^\iota_n[\![\tau_1]\!]\rho$.
- Consequently, $(k - j_1 - j_2, w'', e'_1[v_3/x], e'_2[v_4/x]) \in E^\iota_n[\![\tau_2]\!]\rho$.
- By Closure Under World Extension (2.8), $(k - j_1 - j_2 - 1, \lfloor w'' \rfloor_{k-j_1-j_2-1}, e'_1[v_3/x], e'_2[v_4/x]) \in E^\iota_n[\![\tau_2]\!]\rho$.
- Therefore there exists $(k-j, w''') \sqsupseteq (k-j_1-j_2-1, \lfloor w'' \rfloor_{k-j_1-j_2-1})$ such that $w''.\sigma_2; e'_2[v_4/x] \hookrightarrow^* w'''.\sigma_2; v_2$ and $w'''.\sigma_1 = \sigma_1$ and $(k - j, w''', v_1, v_2) \in V^\iota_n[\![\tau_2]\!]\rho$.

■

Lemma 2.21 (Compatibility: gen)
If $\Delta, \alpha; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau$, then $\Delta; \Gamma \vdash \Lambda\alpha.e_1 \precsim^\iota \Lambda\alpha.e_2 : \forall\alpha.\tau$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(\Lambda\alpha.e_1), \delta_2\gamma_2(\Lambda\alpha.e_2)) \in E^\iota_n[\![\tau]\!]\rho$
- By Inclusion (2.3) it suffices to show $(k, w, \delta_1\gamma_1(\Lambda\alpha.e_1), \delta_2\gamma_2(\Lambda\alpha.e_2)) \in V^\iota_n[\![\tau]\!]\rho$.
- So suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T^{\neg\iota}_{k'}[\![\Omega]\!]w'$.
- To show: $(k'', w'', \delta_1\gamma_1(e_1)[\tau_1/\alpha], \delta_2\gamma_2(e_2)[\tau_2/\alpha]) \in E^\iota_n[\![\tau]\!]\rho, \alpha \mapsto r$
- By assumption and Lemma 2.7, $(\delta_1, \delta_2, \lfloor \rho \rfloor_{k'}) \in D^\iota_{k'}[\![\Delta]\!]w'$.
- Since $\neg\iota = \iota$ we have $(\tau_1, \tau_2, r) \in T^\iota_{k'}[\![\Omega]\!]w'$.
- Let $(\delta'_1, \delta'_2, \rho') := ((\delta_1, \alpha \mapsto \tau_1), (\delta_2, \alpha \mapsto \tau_2), (\lfloor \rho \rfloor_{k'}, \alpha \mapsto r))$, so $(\delta'_1, \delta'_2, \rho') \in D^\iota_{k'}[\![\Delta, \alpha]\!]w'$.
- Furthermore, $(k'', w'', \gamma_1, \gamma_2) \in G^\iota_{k'}[\![\Gamma]\!]\rho'$ by Closure Under World Extension (2.8), Restriction (2.4), and Irrelevance (2.6).
- Now instantiate the assumption with $(\delta'_1, \delta'_2, \rho')$ and $(k'', w'', \gamma_1, \gamma_2)$ to get $(k'', w'', \delta'_1\gamma_1(e_1), \delta'_2\gamma_2(e_2)) \in E^\iota_{k'}[\![\tau]\!]\rho'$.
- Note that $\delta'_i\gamma_i(e_i) = \delta_i\gamma_i(e_i)[\tau_i/\alpha]$.
- The claim then follows by Irrelevance (2.6) and Restriction (2.4).

■

Lemma 2.22 (Compatibility: inst)
If $\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \forall\alpha.\tau_1$ and $\Delta \vdash \tau_2$, then $\Delta; \Gamma \vdash e_1\, \tau_2 \precsim^\iota e_2\, \tau_2 : \tau_1[\tau_2/\alpha]$.

Proof:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(e_1\,\tau_2), \delta_2\gamma_2(e_2\,\tau_2)) \in E^\iota_n[\![\tau_1[\tau_2/\alpha]]\!]\rho$
- Assume $w.\sigma_1; \delta_1\gamma_1(e_1\,\tau_2)$ terminates in $j_1 + 1 + j_2 =: j < k$ steps:

$$\begin{array}{ll} & w.\sigma_1; \delta_1\gamma_1(e_1\,\tau_2) \\ \hookrightarrow^{j_1} & \sigma_1'; (\Lambda\alpha.e_1')\,\delta_1(\tau_2) \\ \hookrightarrow^{1} & \sigma_1'; e_1'[\delta_1(\tau_2)/\alpha] \\ \hookrightarrow^{j_2} & \sigma_1; v_1 \end{array}$$

- Instantiate the assumption to get $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E^\iota_n[\![\forall\alpha.\tau_1]\!]\rho$.
- Consequently, there is $(k-j_1, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(e_2\,\tau_2) \hookrightarrow^* w'.\sigma_2; (\Lambda\alpha.e_2')\,\delta_2(\tau_2)$ with $w'.\sigma_1 = \sigma_1'$ and $(k - j_1, w', \Lambda\alpha.e_1', \Lambda\alpha.e_2') \in V^\iota_n[\![\forall\alpha.\tau_1]\!]\rho$.
- Let $r := (w'.\sigma_1^*(\delta_1(\tau_2)), w'.\sigma_2^*(\delta_2(\tau_2)), V^\iota_{k-j_1}[\![\tau_2]\!]\rho)$.
- If $\iota = \circ$, then $(\delta_1(\tau_2), \delta_2(\tau_2), r) \in T^\circ_{k-j_1}[\![\Omega]\!]w' = T^{\neg\iota}_{k-j_1}[\![\Omega]\!]w'$ is obvious.
- If $\iota = \bullet$, then we show $(\delta_1(\tau_2), \delta_2(\tau_2), r) \in T^\bullet_{k-j_1}[\![\Omega]\!]w' = T^{\neg\iota}_{k-j_1}[\![\Omega]\!]w'$ as follows:
  - Let $\delta := \mathrm{au}(\delta_1, \delta_2, w_0.\eta)$.
  - It suffices to show $(\delta_1(\tau_2), \delta_2(\tau_2), r) = (w'.\eta^1\delta(\tau_2), w'.\eta^2\delta(\tau_2), (w'.\rho^1\delta(\tau_2), w'.\rho^2\delta(\tau_2), V^\bullet_{k-j_1}[\![\delta(\tau_2)]\!]w'.\rho))$.
  - By Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k-j_1}) \in D^\bullet_{k-j_1}[\![\Delta]\!]w'$.
  - First, $\delta_i(\tau_2) = w'.\eta^i\delta(\tau_2)$ by Lemma 2.13.
  - Second, $w'.\sigma_i^*(\delta_i(\tau_2)) = w'.\sigma_i^*(w'.\eta^i\delta(\tau_2)) = w'.\rho^i\delta(\tau_2)$ because of Lemma 2.13 and $w' \in$ World.
  - Finally, $V^\bullet_{k-j_1}[\![\tau_2]\!]\rho = V^\bullet_{k-j_1}[\![\delta(\tau_2)]\!]w'.\rho$ by Lemma 2.14.
- Instantiating $(k - j_1, w', \Lambda\alpha.e_1', \Lambda\alpha.e_2') \in V^\iota_n[\![\forall\alpha.\tau_1]\!]\rho$ with $(k - j_1 - 1, \lfloor w'\rfloor_{k-j_1-1}) \sqsupset (k - j_1, w') \sqsupseteq (k-j_1, w')$ and $(\delta_1(\tau_2), \delta_2(\tau_2), r)$ yields $(k-j_1-1, \lfloor w'\rfloor_{k-j_1-1}, e_1'[\delta_1(\tau_2)/\alpha], e_2'[\delta_2(\tau_2)]) \in E^\iota_n[\![\tau_1]\!]\rho, \alpha{\mapsto}r$.
- Hence there exists $(k - j, w'') \sqsupseteq (k - j_1 - 1, \lfloor w'\rfloor_{k-j_1-1})$ such that $w'.\sigma_2; e_2'[\delta_2(\tau_2)/\alpha] \hookrightarrow^* w''.\sigma_2; v_2$ with $w''.\sigma_1 = \sigma_1$ and $(k - j, w'', v_1, v_2) \in V^\iota_n[\![\tau_1]\!]\rho, \alpha{\mapsto}r$.
- It remains to show $(k - j, w'', v_1, v_2) \in V^\iota_n[\![\tau_1[\tau_2/\alpha]]\!]\rho$.
- This follows by Restriction (2.4), Substitution (2.10), and Lemma 2.2.

■

Lemma 2.23 (Compatibility: pack)
If $\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau[\tau'/\alpha]$ and $\Delta \vdash \tau'$, then $\Delta; \Gamma \vdash \mathsf{pack}\langle\tau', e_1\rangle\ \mathsf{as}\ \exists\alpha.\tau \precsim^\iota \mathsf{pack}\langle\tau', e_2\rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(\mathsf{pack}\langle\tau', e_1\rangle \text{ as } \exists\alpha.\tau), \delta_2\gamma_2(\mathsf{pack}\langle\tau', e_2\rangle \text{ as } \exists\alpha.\tau)) \in E^\iota_n[\![\exists\alpha.\tau]\!]\rho$
- Assume $w.\sigma_1; \delta_1\gamma_1(\mathsf{pack}\langle\tau', e_1\rangle) \hookrightarrow^j \sigma_1; \mathsf{pack}\langle\delta_1(\tau'), v_1\rangle$ where $j < k$.
- Instantiating the assumption yields $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$.
- Consequently, there exists $(k - j, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(\mathsf{pack}\langle\tau', e_2\rangle) \hookrightarrow^* w'.\sigma_2; \mathsf{pack}\langle\delta_2(\tau'), v_2\rangle$ with $w'.\sigma_1 = \sigma_1$ and $(k - j, w', v_1, v_2) \in V^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$.
- It remains to show $(k - j, w', \mathsf{pack}\langle\delta_1(\tau'), v_1\rangle, \mathsf{pack}\langle\delta_2(\tau'), v_2\rangle) \in V^\iota_n[\![\exists\alpha.\tau]\!]\rho$.
- Let $r := (w'.\sigma_1^*(\delta_1(\tau')), w'.\sigma_2^*(\delta_2(\tau')), V^\circ_{k-j}[\![\tau']\!]\rho)$.
- If $\iota = \circ$, then $(\delta_1(\tau'), \delta_2(\tau'), r) \in T^\iota_{k-j}[\![\Omega]\!]w'.\rho$ is obvious.
- If $\iota = \bullet$, then we show $(\delta_1(\tau'), \delta_2(\tau'), r) \in T^\iota_{k-j}[\![\Omega]\!]w'.\rho$ as follows:
  - Let $\delta := \text{au}(\delta_1, \delta_2, w_0.\eta)$.
  - It suffices to show $(\delta_1(\tau'), \delta_2(\tau'), r) = (w'.\eta^1\delta(\tau'), w'.\eta^2(\tau'), (w'.\rho^1\delta(\tau'), w'.\rho^2\delta(\tau'), V^\bullet_{k-j}[\![\delta(\tau')]\!]w'.\rho))$.
  - By Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k-j}) \in D^\bullet_{k-j}[\![\Delta]\!]w'$.
  - First, $\delta_i(\tau') = w'.\eta^i\delta(\tau')$ by Lemma 2.13.
  - Second, $w'.\sigma_i^*(\delta_i(\tau')) = w'.\sigma_i^*(w'.\eta^i\delta(\tau')) = w'.\rho^i\delta(\tau')$ because of Lemma 2.13 and $w' \in$ World.
  - Finally, $V^\bullet_{k-j}[\![\tau']\!]\rho = V^\bullet_{k-j}[\![\delta(\tau')]\!]w'.\rho$ by Lemma 2.14.
- We claim that $(k'', w'', v_1, v_2) \in V^\iota_n[\![\tau]\!]\rho, \alpha \mapsto r$ for any $(k'', w'') \sqsupset (k - j, w')$.
- By Closure Under World Extension (2.8) we have $(k'', w'', v_1, v_2) \in V^\iota_n[\![\tau[\tau'/\alpha]]\!]\rho$.
- The claim then follows by Restriction (2.4), Substitution (2.10), and Lemma 2.2.

■

Lemma 2.24 (Compatibility: unpack)
If $\Delta; \Gamma \vdash e_1 \precsim^\iota e_2 : \exists\alpha.\tau'$ and $\Delta, \alpha; \Gamma, x{:}\tau' \vdash e_3 \precsim^\iota e_4 : \tau$ with $\Delta \vdash \tau$, then $\Delta; \Gamma \vdash \mathsf{unpack}\langle\alpha, x\rangle{=}e_1 \text{ in } e_3 \precsim^\iota \mathsf{unpack}\langle\alpha, x\rangle{=}e_2 \text{ in } e_4 : \tau$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D^\iota_n[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G^\iota_n[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(\mathsf{unpack}\langle\alpha, x\rangle{=}e_1 \text{ in } e_3), \delta_2\gamma_2(\mathsf{unpack}\langle\alpha, x\rangle{=}e_2 \text{ in } e_4))$ in $E^\iota_n[\![\tau]\!]\rho$
- Assume that $w.\sigma_1; \delta_1\gamma_1(\mathsf{unpack}\langle\alpha, x\rangle{=}e_1 \text{ in } e_3)$ terminates in $j_1 + 1 + j_2 =: j < k$ steps:

$$\begin{array}{ll} & w.\sigma_1; \delta_1\gamma_1(\mathsf{unpack}\langle\alpha, x\rangle{=}e_1 \text{ in } e_3) \\ \hookrightarrow^{j_1} & \sigma_1'; \mathsf{unpack}\langle\alpha, x\rangle{=}\,\mathsf{pack}\langle\tau_1, v_1\rangle \text{ in } \delta_1\gamma_1(e_3) \\ \hookrightarrow^{1} & \sigma_1'; \delta_1\gamma_1(e_3)[\tau_1/\alpha][v_1/x] \\ \hookrightarrow^{j_2} & \sigma_1; v_3 \end{array}$$

- Instantiating the first assumption yields the existence of $(k-j_1, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2\gamma_2(\mathsf{unpack}\langle\alpha, x\rangle{=}e_2 \text{ in } e_4) \hookrightarrow^* w'.\sigma_2; \mathsf{unpack}\langle\alpha, x\rangle{=}\,\mathsf{pack}\langle\tau_2, v_2\rangle \text{ in } \delta_2\gamma_2(e_4)$ with $w'.\sigma_1 = \sigma_1'$ and $(k-j_1, w', \mathsf{pack}\langle\tau_1, v_1\rangle, \mathsf{pack}\langle\tau_2, v_2\rangle) \in V_n^\iota[\![\exists\alpha.\tau']\!]\rho$.
- Hence there is r such that $(\tau_1, \tau_2, r) \in T_{k-j_1}^\iota[\![\Omega]\!]w'$ and $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, v_1, v_2) \in V_n^\iota[\![\tau']\!]\rho, \alpha{\mapsto}r$.
- By Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k-j_1}) \in D_{k-j_1}^\iota[\![\Delta]\!]w'$.
- Let $(\delta_1', \delta_2', \rho') := ((\delta_1, \alpha{\mapsto}\tau_1), ((\delta_2, \alpha{\mapsto}\tau_2), (\lfloor\rho\rfloor_{k-j_1}, \alpha{\mapsto}r)))$, so $(\delta_1', \delta_2', \rho') \in D_{k-j_1}^\iota[\![\Delta, \alpha]\!]w'$.
- By Closure Under World Extension (2.8) we know $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$.
- By Restriction (2.4) and Irrelevance (2.6), $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, \gamma_1, \gamma_2) \in G_{k-j_1}^\iota[\![\Gamma]\!]\rho'$.
- Let $\gamma_i' := \gamma_i, x{\mapsto}v_i$, so by Restriction (2.4) and Irrelevance (2.6) we get $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, \gamma_1', \gamma_2') \in G_{k-j_1}^\iota[\![\Gamma, x{:}\tau'']\!]\rho'$.
- Now, instantiating the second assumption with $w' \in \text{World}_{k-j_1}$, $(\delta_1', \delta_2', \rho') \in D_{k-j_1}^\iota[\![\Delta, \alpha]\!]w'$ and $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, \gamma_1', \gamma_2') \in G_{k-j_1}^\iota[\![\Gamma, x{:}\tau'']\!]\rho'$ yields $(k-j_1-1, \lfloor w' \rfloor_{k-j_1-1}, \delta_1'\gamma_1'(e_3), \delta_2'\gamma_2'(e_4)) \in E_{k-j_1}^\iota[\![\tau]\!]\rho'$.
- Note that

$$\begin{array}{ll}
\delta_i'\gamma_i'(e_{i+2}) & \\
= \delta_i(\gamma_i(e_{i+2})[v_i/x])[\tau_i/\alpha]) & \\
= \delta_i\gamma_i(e_{i+2})[v_i/x][\tau_i/\alpha]) & \text{since } \vdash w'.\sigma_i; v_i : (\rho, \alpha{\mapsto}V_{k-j_1}^\iota[\![\tau'']\!]w'.\rho)^i(\tau') \\
= \delta_i\gamma_i(e_{i+2})[\tau_i/\alpha][v_i/x] & \text{since } \vdash w'.\sigma_i; v_i : (\rho, \alpha{\mapsto}V_{k-j_1}^\iota[\![\tau'']\!]w'.\rho)^i(\tau')
\end{array}$$

- Therefore, $\sigma_1'; \delta_1\gamma_1(e_3)[\tau_1/\alpha][v_1/x] \hookrightarrow^{j_2} \sigma_1; v_3$ implies the existence of $(k-j, w'') \sqsupseteq (k-j_1-1, \lfloor w' \rfloor_{k-j_1})$ such that $w'.\sigma_2; \delta_2\gamma_2(e_4)[\tau_2/\alpha][v_2/x] \hookrightarrow^* w''\sigma_2; v_4$ with $w''.\sigma_1 = \sigma_1$ and $(k-j, w'', v_3, v_4) \in V_{k-j_1}^\iota[\![\tau]\!]\rho'$.
- By Restriction (2.4) and, since $\Delta \vdash \tau$, by Irrelevance (2.6), $(k-j, w'', v_3, v_4) \in V_n^\iota[\![\tau]\!]\rho$.

■

Lemma 2.25 (Compatibility: new)
If $\Delta, \alpha{\approx}\tau'; \Gamma \vdash e_1 \precsim^\iota e_2 : \tau$ and $\Delta \vdash \tau$ and $\Delta \vdash \Gamma$, then $\Delta; \Gamma \vdash \mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_1 \precsim^\iota \mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_2 : \tau$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\iota[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(\mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_1), \delta_2\gamma_2(\mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_2)) \in E_n^\iota[\![\tau]\!]\rho$.
- Assume $w.\sigma_1; \delta_1\gamma_1(\mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_1)$ terminates in $1 + j' =: j < k$ steps:

$$\begin{array}{ll}
 & w.\sigma_1; \delta_1\gamma_1(\mathsf{new}\,\alpha{\approx}\tau' \text{ in } e_1) \\
\hookrightarrow^1 & w.\sigma_1, \alpha_1{\approx}\delta_1(\tau'); \delta_1\gamma_1(e_1)[\alpha_1/\alpha] \\
\hookrightarrow^{j'} & \sigma_1; v_1
\end{array}$$

- Note that $w.\sigma_2; \delta_2\gamma_2(\textsf{new}\, \alpha{\approx}\tau' \textsf{ in } e_2) \hookrightarrow^1 w.\sigma_2, \alpha_2{\approx}\delta_2(\tau'); \delta_2\gamma_2(e_2)[\alpha_2/\alpha]$.
- Let $w_\alpha := w \uplus (\alpha_1{\approx}\delta_1(\tau'), \alpha_2{\approx}\delta_2(\tau'), \alpha{\mapsto}(\alpha_1, \alpha_2), \alpha{\mapsto}(\rho^1(\tau'), \rho^2(\tau'), V_k^\iota[\![\tau']\!]\lfloor\rho\rfloor_k))$, so $(k, w_\alpha) \sqsupseteq (k, w)$.
- By Lemma 2.7, $(\delta_1, \delta_2, \lfloor\rho\rfloor_k) \in D_k^\iota[\![\Delta]\!]w_\alpha$.
- Let $(\delta_1', \delta_2', \rho') := ((\delta_1, \alpha{\mapsto}\alpha_1), (\delta_2, \alpha{\mapsto}\alpha_2), (\lfloor\rho\rfloor_k, \alpha{\mapsto}(\rho^1(\tau'), \rho^2(\tau'), V_k^\iota[\![\tau']\!]\lfloor\rho\rfloor_k)))$.
- Note that $w_\alpha.\sigma_i(\alpha_i) = \delta_i(\tau')$, $\alpha_i = w_\alpha.\eta^i(\alpha)$, and $w_\alpha.\rho(\alpha).R = V_k^\iota[\![\tau']\!]\lfloor\rho\rfloor_k$.
- Therefore, $(\delta_1', \delta_2', \rho') \in D_k^\iota[\![\Delta, \alpha{\approx}\tau']\!]w_\alpha$.
- By Closure Under World Extension (2.8) we know $(k-1, \lfloor w_\alpha\rfloor_{k-1}, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$.
- By Restriction (2.4) and Irrelevance (2.6), $(k-1, \lfloor w_\alpha\rfloor_{k-1}, \gamma_1, \gamma_2) \in G_k^\iota[\![\Gamma]\!]\rho'$.
- Now instantiate the assumption with $w_\alpha \in \text{World}_k$, $(\delta_1', \delta_2', \rho') \in D_k^\iota[\![\Delta, \alpha{\approx}\tau']\!]w_\alpha$, $(k-1, \lfloor w_\alpha\rfloor_{k-1}, \gamma_1, \gamma_2) \in G_k^\iota[\![\Gamma]\!]\rho'$ and $(k-1, \lfloor w_\alpha\rfloor_{k-1}) \sqsupset (k, w_\alpha)$ to get $(k-1, \lfloor w\rfloor_{k-1}, \delta_1'\gamma_1(e_1), \delta_2'\gamma_2(e_2)) \in E_k^\iota[\![\tau]\!]\rho'$.
- Note that $\delta_i'\gamma_i(e_i) = \delta_i\gamma_i(e_i)[\alpha_i/\alpha]$.
- Consequently, there exists $(k-j, w'') \sqsupseteq (k-1, w_\alpha)$ such that $w.\sigma_2, \alpha_2{\approx}\delta_2(\tau'); \delta_2\gamma_2(e_2) \hookrightarrow^* w''.\sigma_2; v_2$ with $w''.\sigma_1 = \sigma_1$ and $(k-j, w'', v_1, v_2) \in V_k^\iota[\![\tau]\!]\rho'$.
- Because of $\Delta \vdash \tau$, Irrelevance (2.6) and Restriction (2.4) yield $(k-j, w'', v_1, v_2) \in V_n^\iota[\![\tau]\!]\rho$.

■

Lemma 2.26 (Compatibility: cast)
If $\Delta \vdash \Gamma$ and $\Delta \vdash \tau_1$ and $\Delta \vdash \tau_2$, then $\Delta; \Gamma \vdash \textsf{cast}\ \tau_1\, \tau_2 \precsim^\bullet \textsf{cast}\ \tau_1\, \tau_2 : \tau_1 \to \tau_2 \to \tau_2$.

Proof:

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\bullet[\![\Gamma]\!]\rho$ and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \textsf{cast}\ \delta_1(\tau_1)\, \delta_1(\tau_2), \textsf{cast}\ \delta_2(\tau_1)\, \delta_2(\tau_2)) \in E_n^\bullet[\![\tau_1 \to \tau_2 \to \tau_2]\!]\rho$.
- Case $\delta_1(\tau_1) = \delta_1(\tau_2)$:
  - Then we have the following reductions: $w.\sigma_i; \textsf{cast}\ \delta_i(\tau_1)\, \delta_i(\tau_2) \hookrightarrow^1 w.\sigma_i; \lambda x_1{:}\delta_i(\tau_1).\lambda x_2{:}\delta_i(\tau_2).x_1$
  - Hence it suffices to show $(k-1, \lfloor w\rfloor_{k-1}, \lambda x_1{:}\delta_1(\tau_1).\lambda x_2{:}\delta_1(\tau_2).x_1, \lambda x_1{:}\delta_2(\tau_1).\lambda x_2{:}\delta_2(\tau_2).x_1) \in V_n^\bullet[\![\tau_1 \to \tau_2 \to \tau_2]\!]\rho$.
  - So suppose $(k', w') \sqsupseteq (k-1, \lfloor w\rfloor_{k-1})$ and $(k', w', v_1, v_2) \in V_n^\bullet[\![\tau_1]\!]\rho$.
  - To show: $(k', w', \lambda x_2{:}\delta_1(\tau_2).v_1, \lambda x_2{:}\delta_2(\tau_2).v_2) \in E_n^\bullet[\![\tau_2 \to \tau_2]\!]\rho$.
  - By Inclusion (2.3) it suffices to show $(k', w', \lambda x_2{:}\delta_1(\tau_2).v_1, \lambda x_2{:}\delta_2(\tau_2).v_2) \in V_n^\bullet[\![\tau_2 \to \tau_2]\!]\rho$.
  - So suppose $(k'', w'') \sqsupseteq (k', w')$ and $(k'', w'', v_1', v_2') \in V_n^\bullet[\![\tau_2]\!]\rho$.
  - To show: $(k'', w'', v_1, v_2) \in E_n^\bullet[\![\tau_2]\!]$

  - By Inclusion (2.3) it suffices to show $(k'', w'', v_1, v_2) \in V_n^\bullet[\![\tau_2]\!]\rho$.
  - By Closure Under World Extension (2.8), $(k'', w'', v_1, v_2) \in V_n^\bullet[\![\tau_1]\!]\rho$.
  - Let $\delta := \mathrm{au}(\delta_1, \delta_2, w_0.\eta)$.
  - Then $\delta(\tau_1) = w_0.\eta^{-i}\delta_1(\tau_1) = w_0.\eta^{-i}\delta_1(\tau_2) = \delta(\tau_2)$ by Lemma 2.13.
  - The claim then follows by Lemma 2.14.

- Case $\delta_1(\tau_1) \neq \delta_1(\tau_2)$:
  - Then we have the following reductions:
    $w.\sigma_i; \mathsf{cast}\ \delta_i(\tau_1)\,\delta_i(\tau_2) \hookrightarrow^1 w.\sigma_i; \lambda x_1{:}\delta_i(\tau_1).\lambda x_2{:}\delta_i(\tau_2).x_2$
  - Hence it suffices to show
    $(k-1, \lfloor w\rfloor_{k-1}, \lambda x_1{:}\delta_1(\tau_1).\lambda x_2{:}\delta_2(\tau_1).x_2, \lambda x_1{:}\delta_2(\tau_1).\lambda x_2{:}\delta_2(\tau_2).x_2) \in V_n^\bullet[\![\tau_1 \to \tau_2 \to \tau_2]\!]\rho$.
  - So suppose $(k', w') \sqsupseteq (k-1, \lfloor w\rfloor_{k-1})$ and $(k', w', v_1, v_2) \in V_n^\bullet[\![\tau_1]\!]\rho$.
  - To show: $(k', w', \lambda x_2{:}\delta_2(\tau_1).x_2, \lambda x_2{:}\delta_2(\tau_2).x_2) \in E_n^\bullet[\![\tau_2 \to \tau_2]\!]\rho$.
  - By Inclusion (2.3) it suffices to show $(k', w', \lambda x_2{:}\delta_2(\tau_1).x_2, \lambda x_2{:}\delta_2(\tau_2).x_2) \in V_n^\bullet[\![\tau_2 \to \tau_2]\!]\rho$.
  - So suppose $(k'', w'') \sqsupseteq (k', w')$ and $(k'', w'', v_1', v_2') \in V_n^\bullet[\![\tau_2]\!]\rho$.
  - By Inclusion (2.3) it suffices to show $(k'', w'', v_1', v_2') \in V_n^\bullet[\![\tau_2]\!]\rho$, which is given.

■

Lemma 2.27 (Compatibility: conv)
If $\Delta;\Gamma \vdash e_1 \precsim^\iota e_2 : \tau'$ and $\Delta \vdash \tau \approx \tau'$, then $\Delta;\Gamma \vdash e_1 \precsim^\iota e_2 : \tau$.

Proof: Follows from Type Compatibility (2.11). ■

Note that we do *not* have compatibility of $\precsim^\circ$ with CAST.

Lemma 2.28
If $\Delta;\Gamma \vdash e_1 \precsim^\iota e_2 : \tau$, $\Delta' \supseteq \Delta$, $\Gamma' \supseteq \Gamma$, and $\Delta' \vdash \Gamma$, then $\Delta';\Gamma' \vdash e_1 \precsim^\iota e_2 : \tau$.

Proof:

- By assumption and Weakening we know $\Delta';\Gamma' \vdash e_i : \tau$.
- So suppose $w_0 \in \mathrm{World}_n$, $(\delta_1', \delta_2', \rho') \in D_n^\iota[\![\Delta']\!]w_0$, $(k, w, \gamma_1', \gamma_2') \in G_n^\iota[\![\Gamma']\!]\rho'$, and $(k, w) \sqsupseteq (n, w_0)$.
- To show: $(k, w, \delta_1'\gamma_1'(e_1), \delta_2'\gamma_2'(e_2)) \in E_n^\iota[\![\tau]\!]\rho'$
- Let δ_i, ρ, γ_i be the restriction of δ_i', ρ', γ_i' to $\mathrm{dom}(\Delta)$, $\mathrm{dom}(\Delta)$, $\mathrm{dom}(\Gamma)$ respectively.
- It's easy to see that then $(\delta_1, \delta_2, \rho) \in D_n^\iota[\![\Delta]\!]w_0$ and, with the help of Irrelevance (2.6), $(k, w, \gamma_1, \gamma_2) \in G_n^\iota[\![\Gamma]\!]\rho$.
- Hence by assumption, $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E^\iota[\![\tau]\!]\rho$.

- Since $\Delta \vdash \tau$ and $\Delta; \Gamma \vdash \gamma_i$ by assumption and Validity, we have $\delta_i' \gamma_i'(e_i) = \delta_i \gamma_i(e_i)$.
- The claim then follows by Irrelevance (2.6).

■

Lemma 2.29 (Fundamental Property)
If $\Delta; \Gamma \vdash e : \tau$, then $\Delta; \Gamma \vdash e \precsim^\bullet e : \tau$.

Proof: By induction on the derivation, in each case using the appropriate compatibility lemma. ■

Lemma 2.30 (Fundamental Property)
If $\Delta; \Gamma \vdash e : \tau$ and e does not use cast, then $\Delta; \Gamma \vdash e \precsim^\circ e : \tau$.

Proof: By induction on the derivation, in each case using the appropriate compatibility lemma. ■

Lemma 2.31 (Congruence)
If $\Delta; \Gamma \vdash e_1 : \tau$ and $\Delta; \Gamma \vdash e_2 : \tau$ and $\Delta; \Gamma \vdash e_1 \precsim^\bullet e_2 : \tau$ and $C : (\Delta; \Gamma; \tau) \rightsquigarrow (\Delta'; \Gamma'; \tau')$, then $\Delta'; \Gamma' \vdash C[e_1] \precsim^\bullet C[e_2] : \tau'$.

Proof: By induction on the derivation of the context typing, in each case using the appropriate compatibility lemma and possibly the Fundamental Property. For $C = [\_]$ we use Lemma 2.28. ■

Theorem 2.32 (Soundness of $\precsim^\bullet$ wrt. $\preceq$)
If $\Delta; \Gamma \vdash e_1 : \tau$ and $\Delta; \Gamma \vdash e_2 : \tau$ and $\Delta; \Gamma \vdash e_1 \precsim^\bullet e_2 : \tau$, then $\Delta; \Gamma \vdash e_1 \preceq e_2 : \tau$.

Proof:

- Suppose $\vdash \sigma$ and $C : (\Delta; \Gamma; \tau) \rightsquigarrow (\sigma; \emptyset; \tau')$ and $\sigma; C[e_1]\downarrow$, i.e., $\sigma; C[e_1] \hookrightarrow^j \sigma_1; v_1$.
- To show: $\sigma; C[e_2]\downarrow$, i.e., $\sigma; C[e_2] \hookrightarrow^* \sigma_2; v_2$
- By Congruence (2.31) we have $\sigma; \emptyset \vdash C[e_1] \precsim^\bullet C[e_2] : \tau'$.
- Say $\sigma = \alpha_1{\approx}\tau_1, \ldots, \alpha_n{\approx}\tau_n$.
- Let

$$
\begin{aligned}
\sigma_0 &:= \epsilon \\
\sigma_{i+1} &:= \sigma_i, \alpha_{i+1}{\approx}\tau_{i+1} \\
\delta_0 &:= \emptyset \\
\delta_{i+1} &:= \delta_i, \alpha_{i+1}{\mapsto}\alpha_{i+1} \\
\rho_0 &:= \emptyset \\
\rho_{i+1} &:= \rho_i, \alpha_{i+1}{\mapsto}V^\bullet_{j+2}[\![\tau_{i+1}]\!]\rho_i \\
w &:= (\sigma, \sigma, \{\alpha_i{\mapsto}(\alpha_i, \alpha_i) \mid 1 \le i \le n\}, \rho_n)
\end{aligned}
$$

- First, note that $\rho_i \in \text{Interp}_{j+2}$ and $w \in \text{World}_{j+2}$.
- Furthermore, given $0 \leq i < n$, it is easy to see that $(\delta_i, \delta_i, \rho_i) \in D^\bullet_{j+2}[\![\sigma_i]\!]w$ implies $(\delta_{i+1}, \delta_{i+1}, \rho_{i+1}) \in D^\bullet_{j+2}[\![\sigma_{i+1}]\!]w$.
- Together with $(\delta_0, \delta_0, \rho_0) \in D^\bullet_{j+2}[\![\epsilon]\!]w$ this means $(\delta_n, \delta_n, \rho_n) \in D^\bullet_{j+2}[\![\sigma]\!]w$.
- Now instantiate $\sigma; \emptyset \vdash C[e_1] \precsim^\bullet C[e_2] : \tau'$ with $w \in \text{World}_{j+2}$, $(\delta_n, \delta_n, \rho_n) \in D^\bullet_{j+2}[\![\sigma]\!]w$ and $(j, \lfloor w \rfloor_j, \emptyset, \emptyset) \in G^\bullet_{j+2}[\![\epsilon]\!]\rho_n$ to get $(j+1, \lfloor w \rfloor_{j+1}, \delta_n\emptyset(C[e_1]), \delta_n\emptyset(C[e_2])) \in E^\bullet_{j+2}[\![\tau']\!]\rho_n$.
- Note that $\delta_n\emptyset(C[e_i]) = C[e_i]$.
- Consequently, $\sigma; C[e_1] \hookrightarrow^j \sigma_1; v_1$ implies $\sigma; C[e_2] \hookrightarrow^* \sigma_2; v_2$.

■

2.4 Wrapping

$$
\begin{array}{lcl}
\mathsf{new}^{+}\ \alpha\ \mathsf{in}\ e & := & \mathsf{new}\,\alpha{\approx}\alpha\ \mathsf{in}\ e \\
\mathsf{new}^{-}\ \alpha\ \mathsf{in}\ e & := & e
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{Wr}^{\pm}_{\alpha}(v) & \stackrel{\mathrm{def}}{=} & v \\
\mathrm{Wr}^{\pm}_{b}(v) & \stackrel{\mathrm{def}}{=} & v \\
\mathrm{Wr}^{\pm}_{\tau_1\times\tau_2}(v) & \stackrel{\mathrm{def}}{=} & \mathsf{let}\,x{=}v\ \mathsf{in}\ \langle \mathrm{Wr}^{\pm}_{\tau_1}(x.1), \mathrm{Wr}^{\pm}_{\tau_2}(x.2)\rangle \\
\mathrm{Wr}^{\pm}_{\tau_1\to\tau_2}(v) & \stackrel{\mathrm{def}}{=} & \lambda x{:}\tau_1.\, \mathrm{Wr}^{\pm}_{\tau_2}(v\ \mathrm{Wr}^{\mp}_{\tau_1}(x)) \\
\mathrm{Wr}^{\pm}_{\exists\alpha.\tau}(v) & \stackrel{\mathrm{def}}{=} & \mathsf{unpack}\,\langle\alpha,x\rangle{=}v\ \mathsf{in}\ \mathsf{new}^{\pm}\ \alpha\ \mathsf{in}\ \mathsf{pack}\,\langle\alpha, \mathrm{Wr}^{\pm}_{\tau}(x)\rangle \\
\mathrm{Wr}^{\pm}_{\forall\alpha.\tau}(v) & \stackrel{\mathrm{def}}{=} & \Lambda\alpha.\, \mathsf{new}^{\mp}\ \alpha\ \mathsf{in}\ \mathrm{Wr}^{\pm}_{\tau}(v\,\alpha) \\
\mathrm{Wr}^{\pm}_{\tau}(e) & \stackrel{\mathrm{def}}{=} & \mathsf{let}\,x{=}e\ \mathsf{in}\ \mathrm{Wr}^{\pm}_{\tau}(x)
\end{array}
$$

Lemma 2.33
If $\sigma : e \hookrightarrow^{j} \sigma'; e'$, then $\sigma; \mathrm{Wr}^{\pm}_{\tau}(e) \hookrightarrow^{j} \sigma'; \mathrm{Wr}^{\pm}_{\tau}(e')$.

Proof: Obvious from the definition. ■

2.5 Relating the Relations

Lemma 2.34
$T^{\bullet}_{n}[\![\Omega]\!]w = T^{-}_{n}[\![\Omega]\!]w \subseteq T^{+}_{n}[\![\Omega]\!]w = T^{\circ}_{n}[\![\Omega]\!]w$

Theorem 2.35 (Chain)

1. $V^{-}_{n}[\![\tau]\!]\rho \subseteq V^{\iota}_{n}[\![\tau]\!]\rho \subseteq V^{+}_{n}[\![\tau]\!]\rho$
2. $E^{-}_{n}[\![\tau]\!]\rho \subseteq E^{\iota}_{n}[\![\tau]\!]\rho \subseteq E^{+}_{n}[\![\tau]\!]\rho$

Proof: By mutual induction on τ.

1.
   - Case $\tau = \alpha$: Trivial
   - Case $\tau = \tau' \times \tau''$: Follows easily by induction.
   - Case $\tau = \tau'' \to \tau'$:
     (a) $V^{-}_{n}[\![\tau]\!]\rho \subseteq V^{\iota}_{n}[\![\tau]\!]\rho$:
       - Suppose $(k, w, \lambda x{:}\tau_1.e_1, \lambda x{:}\tau_2.e_2) \in V^{-}_{n}[\![\tau'' \to \tau']\!]\rho$.
       - Suppose $(k', w', v_1, v_2) \in V^{-\iota}_{n}[\![\tau'']\!]\rho$ with $(k', w') \sqsupseteq (k, w)$.
       - By induction, $(k', w', v_1, v_2) \in V^{+}_{n}[\![\tau'']\!]\rho$.
       - Hence by assumption, $(k', w', e_1[v_1/x], e_2[v_2/x]) \in E^{-}_{n}[\![\tau']\!]\rho$.
       - So by induction, $(k', w', e_1[v_1/x], e_2[v_2/x]) \in E^{\iota}_{n}[\![\tau']\!]\rho$.
       - Therefore, $(k, w, \lambda x{:}\tau_1.e_1, \lambda x{:}\tau_2.e_2) \in V^{\iota}_{n}[\![\tau'' \to \tau']\!]\rho$.

     (b) $V^{\iota}_{n}[\![\tau]\!]\rho \subseteq V^{+}_{n}[\![\tau]\!]\rho$:
       - Suppose $(k, w, \lambda x{:}\tau_1.e_1, \lambda x{:}\tau_2.e_2) \in V^{\iota}_{n}[\![\tau'' \to \tau']\!]\rho$.
       - Suppose $(k', w', v_1, v_2) \in V^{-}_{n}[\![\tau'']\!]\rho$ with $(k', w') \sqsupseteq (k, w)$.

- By induction, $(k', w', v_1, v_2) \in V_n^{\iota}[\![\tau'']\!]\rho$.
- Hence by assumption, $(k', w', e_1[v_1/x], e_2[v_2/x]) \in E_n^{\iota}[\![\tau']\!]\rho$.
- So by induction, $(k', w', e_1[v_1/x], e_2[v_2/x]) \in E_n^{+}[\![\tau']\!]\rho$.
- Therefore, $(k, w, \lambda x{:}\tau_1.e_1, \lambda x{:}\tau_2.e_2) \in V_n^{+}[\![\tau'' \to \tau']\!]\rho$.

- Case $\tau = \forall\alpha.\tau'$:

  (a) $V_n^{-}[\![\tau]\!]\rho \subseteq V_n^{\iota}[\![\tau]\!]\rho$:
  - Suppose $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^{-}[\![\forall\alpha.\tau']\!]\rho$.
  - Suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T_{k'}^{\neg\iota}[\![\Omega]\!]w'$.
  - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T_{k'}^{+}[\![\Omega]\!]w'$.
  - Hence by assumption, $(k', w', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^{-}[\![\tau']\!]\rho, \alpha \mapsto r$.
  - So by induction, $(k', w', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^{\iota}[\![\tau']\!]\rho, \alpha \mapsto r$.
  - Therefore, $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^{\iota}[\![\forall\alpha.\tau']\!]\rho$.

  (b) $V_n^{\iota}[\![\tau]\!]\rho \subseteq V_n^{+}[\![\tau]\!]\rho$:
  - Suppose $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^{\iota}[\![\forall\alpha.\tau']\!]\rho$.
  - Suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T_{k'}^{-}[\![\Omega]\!]w'$.
  - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T_{k'}^{\iota}[\![\Omega]\!]w'$.
  - Hence by assumption, $(k', w', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^{\iota}[\![\tau']\!]\rho, \alpha \mapsto r$.
  - So by induction, $(k', w', e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^{+}[\![\tau']\!]\rho, \alpha \mapsto r$.
  - Therefore, $(k, w, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^{+}[\![\forall\alpha.\tau']\!]\rho$.

- Case $\tau = \exists\alpha.\tau'$:

  (a) $V_n^{-}[\![\tau]\!]\rho \subseteq V_n^{\iota}[\![\tau]\!]\rho$:
  - Suppose $(k, w, \mathsf{pack}\langle\tau_1, v_1\rangle, \mathsf{pack}\langle\tau_2, v_2\rangle) \in V_n^{-}[\![\exists\alpha.\tau']\!]\rho$.
  - So there is r such that $(\tau_1, \tau_2, r) \in T_k^{-}[\![\Omega]\!]w$ and $(k', w', v_1, v_2) \in V_n^{-}[\![\tau']\!]\rho, \alpha \mapsto r$ for any $(k', w') \sqsupset (k, w)$.
  - So by induction, $(k', w', v_1, v_2) \in V_n^{\iota}[\![\tau']\!]\rho, \alpha \mapsto r$ for any $(k', w') \sqsupset (k, w)$.
  - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T_k^{\iota}[\![\Omega]\!]w$.
  - Therefore, $(k, w, \mathsf{pack}\langle\tau_1, e_1\rangle, \mathsf{pack}\langle\tau_2, e_2\rangle) \in V_n^{\iota}[\![\exists\alpha.\tau']\!]\rho$.

  (b) $V_n^{\iota}[\![\tau]\!]\rho \subseteq V_n^{+}[\![\tau]\!]\rho$:
  - Suppose $(k, w, \mathsf{pack}\langle\tau_1, v_1\rangle, \mathsf{pack}\langle\tau_2, v_2\rangle) \in V_n^{\iota}[\![\exists\alpha.\tau']\!]\rho$.
  - So there is r such that $(\tau_1, \tau_2, r) \in T_k^{\iota}[\![\Omega]\!]w$ and $(k', w', v_1, v_2) \in V_n^{\iota}[\![\tau']\!]\rho, \alpha \mapsto r$ for any $(k', w') \sqsupset (k, w)$.
  - So by induction, $(k', w', v_1, v_2) \in V_n^{+}[\![\tau']\!]\rho, \alpha \mapsto r$ for any $(k', w') \sqsupset (k, w)$.
  - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T_k^{+}[\![\Omega]\!]w$.
  - Therefore, $(k, w, \mathsf{pack}\langle\tau_1, e_1\rangle, \mathsf{pack}\langle\tau_2, e_2\rangle) \in V_n^{+}[\![\exists\alpha.\tau']\!]\rho$.

2. Follows easily from part (1).

■

Theorem 2.36 (Wrapping I)
Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^{\circ}[\![\Delta]\!]w_0$, $(k, w) \sqsupset (n, w_0)$, and $\Delta \vdash \tau$.

1. (a) If $(k, w, v_1, v_2) \in V_n^{\circ,+}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^+(v_1)), \delta_2(\mathrm{Wr}_\tau^+(v_2))) \in E_n^{-,\bullet}[\![\tau]\!]\rho$.

   (b) If $(k, w, e_1, e_2) \in E_n^{\circ,+}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^+(e_1)), \delta_2(\mathrm{Wr}_\tau^+(e_2))) \in E_n^{-,\bullet}[\![\tau]\!]\rho$.

2. (a) If $(k, w, v_1, v_2) \in V_n^{+,\bullet}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^-(v_1)), \delta_2(\mathrm{Wr}_\tau^-(v_2))) \in E_n^{\circ,-}[\![\tau]\!]\rho$.

   (b) If $(k, w, e_1, e_2) \in E_n^{+,\bullet}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^-(e_1)), \delta_2(\mathrm{Wr}_\tau^-(e_2))) \in E_n^{\circ,-}[\![\tau]\!]\rho$.

Proof: By induction on τ. Note that δ_i only affects the type annotations of abstractions and packages produced by $\mathrm{Wr}^\pm$.

1. (a)
   - Case $\tau = \alpha$ or $\tau = b$: Trivial.
   - Case $\tau = \tau' \to \tau''$: $v_i = \lambda x{:}\tau_i.e_i$
     - Suppose $(k', w', v_3, v_4) \in V_n^{+,\bullet}[\![\tau']\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.
     - To show: $(k', w', \delta_1(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_1.e_1)\ \mathrm{Wr}_{\tau'}^-(v_3))), \delta_2(\mathrm{Wr}_{\tau''}^+((\lambda x.e_2)\ \mathrm{Wr}_{\tau'}^-(v_4)))) \in E_n^{-,\bullet}[\![\tau'']\!]\rho$.
     - So suppose $w'.\sigma_1; \delta_1(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_1.e_1)\ \mathrm{Wr}_{\tau'}^-(v_3)))$ terminates in $j_1 + 1 + j_2 =: j < k$ steps:
     $$\begin{array}{ll} & w'.\sigma_1; \delta_1(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_1.e_1)\ \mathrm{Wr}_{\tau'}^-(v_3))) \\ \hookrightarrow^{j_1} & \sigma_1''; \delta_1(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_1.e_1)\, v_3')) \\ \hookrightarrow^{1} & \sigma_1''; \delta_1(\mathrm{Wr}_{\tau''}^+(e_1[v_3'/x])) \\ \hookrightarrow^{j_2} & \sigma_1; v_1' \end{array}$$
     - By induction, $(k', w', \delta_1(\mathrm{Wr}_{\tau'}^-(v_3)), \delta_2(\mathrm{Wr}_{\tau'}^-(v_4))) \in E_n^{\circ,-}[\![\tau']\!]\rho$.
     - This implies the existence of $(k' - j_1, w'') \sqsupseteq (k', w')$ such that $w'.\sigma_2; \delta_2(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_2.e_2)\ \mathrm{Wr}_{\tau'}^-(v_4))) \hookrightarrow^* w''.\sigma_2; \delta_2(\mathrm{Wr}_{\tau''}^+((\lambda x{:}\tau_2.e_2)\, v_4'))$ with $w''.\sigma_1 = \sigma_1''$ and $(k' - j_1, w'', v_3', v_4') \in V_n^{\circ,-}[\![\tau']\!]\rho$.
     - So by assumption, $(k' - j_1, w'', e_1[v_3'/x], e_2[v_4'/x]) \in E_n^{\circ,+}[\![\tau'']\!]\rho$.
     - By induction, $(k' - j_1, w'', \delta_1(\mathrm{Wr}_{\tau''}^+(e_1[v_3'/x])), \delta_2(\mathrm{Wr}_{\tau''}^+(e_2[v_4'/x]))) \in E_n^{-,\bullet}[\![\tau'']\!]\rho$.
     - Now by Closure Under World Extension (2.8), $(k' - j_1 - 1, \lfloor w'' \rfloor_{k-j_1-1}, \delta_1(\mathrm{Wr}_{\tau''}^+(e_1[v_3'/x])), \delta_2(\mathrm{Wr}_{\tau''}^+(e_2[v_4'/x]))) \in E_n^{-,\bullet}[\![\tau'']\!]\rho$.
     - Hence there exists $(k' - j, w''') \sqsupseteq (k' - j_1 - 1, \lfloor w'' \rfloor_{k-j_1-1})$ such that $w''.\sigma_2; \delta_1(\mathrm{Wr}_{\tau''}^+(e_2[v_4'/x])) \hookrightarrow^* w'''.\sigma_2; v_2'$ with $w'''.\sigma_1 = \sigma_1$ and $(k - j, w''', v_1', v_2') \in V_n^{-,\bullet}[\![\tau'']\!]\rho$.
   - Case $\tau = \forall\alpha.\tau'$: $v_i = \Lambda\alpha.e_i$
     - To show: $(k, w, \Lambda\alpha.\delta_1(\mathrm{Wr}_{\tau'}^+(v_1\,\alpha)), \Lambda\alpha.\delta_2(\mathrm{Wr}_{\tau'}^+(v_2\,\alpha))) \in V_n^{-,\bullet}[\![\forall\alpha.\tau']\!]\rho$
     - Suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T_{k'}^{+,\bullet}[\![\Omega]\!]w'$.
     - To show: $(k'', w'', \delta_1'(\mathrm{Wr}_{\tau'}^+((\Lambda\alpha.e_1)\,\tau_1)), \delta_2'(\mathrm{Wr}_{\tau'}^+((\Lambda\alpha.e_2)\,\tau_2))) \in E_n^{-,\bullet}[\![\tau']\!]\rho, \alpha{\mapsto}r$, for $\delta_i' := \delta_i, \alpha{\mapsto}\tau_i$.
     - So suppose $w''.\sigma_1; \delta_1'(\mathrm{Wr}_{\tau'}^+((\Lambda\alpha.e_1)\,\tau_1))$ terminates in $1 + j_1 + j_2 =: j < k$ steps:
     $$\begin{array}{ll} & w''.\sigma_1; \delta_1'(\mathrm{Wr}_{\tau'}^+((\Lambda\alpha.e_1)\,\tau_1)) \\ \hookrightarrow^{1} & w''.\sigma_1; \delta_1'(\mathrm{Wr}_{\tau'}^+(e_1[\tau_1/\alpha])) \\ \hookrightarrow^{j_1} & \sigma_1'; \delta_1'(\mathrm{Wr}_{\tau'}^+(v_1')) \\ \hookrightarrow^{j_2} & \sigma_1; v_1'' \end{array}$$
     - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T_{k'}^{\circ,-}[\![\Omega]\!]w'$.

- Instantiate the assumption with $(k''-1, \lfloor w'' \rfloor_{k''-1}) \sqsupset (k', w') \sqsupseteq (k, w)$ and (τ_1, τ_2, r) to get $(k''-1, \lfloor w'' \rfloor_{k''-1}, e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in E_n^{\circ,+}[\![\tau']\!]\rho, \alpha{\mapsto}r$.
- Consequently, there exists $(k''-1-j_1, w''') \sqsupseteq (k''-1, \lfloor w'' \rfloor_{k''-1})$ such that $w''.\sigma_2; \delta_2'(\mathrm{Wr}^+_{\tau'}(e_2[\tau_2/\alpha])) \hookrightarrow^* w'''.\sigma_2; \delta_2'(\mathrm{Wr}^+_{\tau'}(v_2'))$ with $w'''.\sigma_1 = \sigma_1'$ and $(k''-1-j_1, w''', v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\rho, \alpha{\mapsto}r$.
- Since $(\delta_1, \delta_2, \lfloor \rho \rfloor_{k'}) \in D^{\circ}_{k'}[\![\Delta, \alpha]\!]w'$ by Lemma 2.7, Lemma 2.34 yields $(\delta_1', \delta_2', (\lfloor \rho \rfloor_{k'}, \alpha{\mapsto}r)) \in D^{\circ}_{k'}[\![\Delta, \alpha]\!]w'$.
- By Restriction (2.4) and Irrelevance (2.6), $(k''-1-j_1, w''', v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\lfloor \rho \rfloor_{k'}, \alpha{\mapsto}r$.
- Hence by induction, $(k''-1-j_1, w''', \delta_1'(\mathrm{Wr}^+_{\tau'}(v_1')), \delta_2'(\mathrm{Wr}^+_{\tau'}(v_2'))) \in E_n^{-,\bullet}[\![\tau']\!]\lfloor \rho \rfloor_{k'}, \alpha{\mapsto}r$.
- Hence there exists $(k''-j, w'''') \sqsupseteq (k''-j_1-1, w''')$ such that $w'''.\sigma_2; \delta_2'(\mathrm{Wr}^+_{\tau'}(v_2')) \hookrightarrow^* w''''.\sigma_2; v_2''$ with $w''''.\sigma_1 = \sigma_1$ and $(k''-j, w'''', v_1'', v_2'') \in V_n^{-,\bullet}[\![\tau']\!]\lfloor \rho \rfloor_{k'}, \alpha{\mapsto}r$.
- By Restriction (2.4) and Irrelevance (2.6) the latter means $(k''-j, w'''', v_1'', v_2'') \in V_n^{-,\bullet}[\![\tau']\!]\rho, \alpha{\mapsto}r$.

- Case $\tau = \exists \alpha.\tau'$: $v_i = \mathsf{pack}\langle \tau_i, v_i' \rangle$
  - To show: $(k, w, \mathsf{unpack}\langle \alpha, x \rangle{=}v_1 \ \mathsf{in\ new}\ \alpha{\approx}\alpha\ \mathsf{in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^+_{\tau'}(x)) \rangle,$ $\mathsf{unpack}\langle \alpha, x \rangle{=}v_2\ \mathsf{in\ new}\ \alpha{\approx}\alpha\ \mathsf{in\ pack}\langle \alpha, \delta_2(\mathrm{Wr}^+_{\tau'}(x)) \rangle) \in E_n^{-,\bullet}[\![\exists \alpha.\tau']\!]\rho$
  - So suppose $w.\sigma_1; \mathsf{unpack}\langle \alpha, x \rangle{=}v_1\ \mathsf{in\ new}\ \alpha{\approx}\alpha\ \mathsf{in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^+_{\tau'}(x)) \rangle$ terminates in $1 + 1 + j' =: j < k$ steps:

$$
\begin{array}{ll}
 & w.\sigma_1; \mathsf{unpack}\langle \alpha, x \rangle{=}v_1\ \mathsf{in\ new}\ \alpha{\approx}\alpha\ \mathsf{in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^+_{\tau'}(x)) \rangle \\
\hookrightarrow^1 & w.\sigma_1; \mathsf{new}\ \alpha{\approx}\tau_1\ \mathsf{in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^+_{\tau'}(v_1')) \rangle \\
\hookrightarrow^1 & w.\sigma_1, \alpha_1{\approx}\tau_1; \mathsf{pack}\langle \alpha_1, \delta_1'(\mathrm{Wr}^+_{\tau'}(v_1')) \rangle \\
\hookrightarrow^{j'} & \sigma_1; \mathsf{pack}\langle \alpha_1, v_1'' \rangle
\end{array}
$$

  where $\delta_1' := \delta_1, \alpha{\mapsto}\alpha_1$
  - Note that

$$
\begin{array}{ll}
 & w.\sigma_2; \mathsf{unpack}\langle \alpha, x \rangle{=}v_2\ \mathsf{in\ new}\ \alpha{\approx}\alpha\ \mathsf{in\ pack}\langle \alpha, \delta_2(\mathrm{Wr}^+_{\tau'}(x)) \rangle \\
\hookrightarrow^1 & w.\sigma_2; \mathsf{new}\ \alpha{\approx}\tau_2\ \mathsf{in\ pack}\langle \alpha, \delta_2(\mathrm{Wr}^+_{\tau'}(v_2')) \rangle \\
\hookrightarrow^1 & w.\sigma_2, \alpha_2{\approx}\tau_2; \mathsf{pack}\langle \alpha_2, \delta_2'(\mathrm{Wr}^+_{\tau'}(v_2')) \rangle
\end{array}
$$

  where $\delta_2' := \delta_2, \alpha{\mapsto}\alpha_2$
  - By assumption we know $(k', w', v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\rho, \alpha{\mapsto}r$ for some r with $(\tau_1, \tau_2, r) \in T_k^{\circ,+}[\![\Omega]\!]w$ and any $(k', w') \sqsupset (k, w)$.
  - Let $w_\alpha := \lfloor w \rfloor_{k-1} \uplus (\alpha_1{\approx}\tau_1, \alpha_2{\approx}\tau_2, \alpha{\mapsto}(\alpha_1, \alpha_2), \alpha{\mapsto}\lfloor r \rfloor_{k-1})$, so $(k-1, w_\alpha) \sqsupset (k, w)$.
  - Hence $(k-1, w_\alpha, v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\rho, \alpha{\mapsto}r$.
  - By Closure Under World Extension (2.8), $(k-2, \lfloor w_\alpha \rfloor_{k-2}, v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\rho, \alpha{\mapsto}r$.
  - Let $r' := (w_\alpha.\rho^1(\alpha), w_\alpha.\rho^2(\alpha), V^{\bullet}_{k-1}[\![\alpha]\!]w_\alpha) = \lfloor r \rfloor_{k-1}$, so $(\alpha_1, \alpha_2, r') \in T^{-,\bullet}_{k-1}[\![\Omega]\!]w_\alpha$.
  - By Restriction (2.4) and Irrelevance (2.6), $(k-2, \lfloor w_\alpha \rfloor_{k-2}, v_1', v_2') \in V_n^{\circ,+}[\![\tau']\!]\rho'$ for $\rho' := \lfloor \rho \rfloor_{k-1}, \alpha{\mapsto}r'$.

– By Lemma 2.34, $(\alpha_1, \alpha_2, r') \in T^{\circ}_{k-1}[\![\Omega]\!]w_\alpha$.
– Furthermore $(\delta_1, \delta_2, \lfloor\rho\rfloor_{k-1}) \in D^{\circ}_{k-1}[\![\Delta]\!]w_\alpha$ by Lemma 2.7, so $(\delta'_1, \delta'_2, \rho') \in D^{\circ}_{k-1}[\![\Delta, \alpha]\!]w_\alpha$.
– Hence induction yields $(k-2, \lfloor w_\alpha\rfloor_{k-2}, \delta'_1(\mathrm{Wr}^+_{\tau'}(v'_1)), \delta'_2(\mathrm{Wr}^+_{\tau'}(v'_2))) \in E^{-,\bullet}_n[\![\tau']\!]\rho'$.
– Because $w_\alpha.\sigma_1 = w.\sigma_1, \alpha_1{\approx}\tau_1$, this implies the existence of $(k-j, w') \sqsupseteq (k-2, w_\alpha)$ such that $w.\sigma_2, \alpha_2{\approx}\tau_2; \mathsf{pack}\langle\alpha_2, \delta'_2(\mathrm{Wr}^+_{\tau'}(v'_2))\rangle \hookrightarrow^* w'.\sigma_2; \mathsf{pack}\langle\alpha_2, v''_2\rangle$ with $w'.\sigma_1 = \sigma_1$ and $(k-j, w', v''_1, v''_2) \in V^{-,\bullet}_n[\![\tau']\!]\rho'$.
– By Restriction (2.4) and Irrelevance (2.6), $(k-j, w', v''_1, v''_2) \in V^{-,\bullet}_n[\![\tau']\!]\rho, \alpha{\mapsto}r'$.
– Since $(\alpha_1, \alpha_2, r') \in T^{-,\bullet}_{k-j}[\![\Omega]\!]w'$ by Lemma 2.34, $(k-j, w', \mathsf{pack}\langle\alpha', v''_1\rangle, \mathsf{pack}\langle\alpha', v''_2\rangle) \in V^{-,\bullet}_n[\![\exists\alpha.\tau']\!]\rho$ follows by Closure Under World Extension (2.8).

(b) • Suppose $w.\sigma_1; \delta_1(\mathrm{Wr}^+_\tau(e_1))$ terminates in $j_1 + j_2 =: j < k$ steps:

$$\begin{array}{rl} & w.\sigma_1; \delta_1(\mathrm{Wr}^+_\tau(e_1)) \\ \hookrightarrow^{j_1} & \sigma'_1; \delta_1(\mathrm{Wr}^+_\tau(v_1)) \\ \hookrightarrow^{j_2} & \sigma_1; v'_1 \end{array}$$

• So by assumption there exists $(k-j_1, w') \sqsupseteq (k, w)$ such that $w.\sigma_2; \delta_2(\mathrm{Wr}^+_\tau(e_2)) \hookrightarrow^* w'.\sigma_2; \delta_2(\mathrm{Wr}^+_\tau(v_2))$ with $w'.\sigma_1 = \sigma'_1$ and $(k-j_1, w', v_1, v_2) \in V^{\circ,+}_n[\![\tau]\!]\rho$.
• By part (a), $(k-j_1, w', \delta_1(\mathrm{Wr}^+_\tau(v_1)), \delta_2(\mathrm{Wr}^+_\tau(v_2))) \in E^{-,\bullet}_n[\![\tau]\!]\rho$.
• Consequently, there exists $(k-j, w'') \sqsupseteq (k-j_1, w')$ such that $w'.\sigma_2; \delta_2(\mathrm{Wr}^+_\tau(v_2)) \hookrightarrow^* w''.\sigma_2; v'_2$ with $w''.\sigma_1 = \sigma_1$ and $(k-j, w'', v'_1, v'_2) \in V^{-,\bullet}_n[\![\tau]\!]\rho$.

2. (a) • Case $\tau = \alpha$ or $\tau = b$: Trivial.
• Case $\tau = \tau' \to \tau''$: Symmetric to arrow case of (1a).
• Case $\tau = \forall\alpha.\tau'$: $v_i = \Lambda\alpha.e_i$
– To show: $(k, w, \Lambda\alpha.\,\mathsf{new}\,\alpha{=}\alpha\ \mathsf{in}\ \delta_1(\mathrm{Wr}^-_{\tau'}(v_1\,\alpha)), \Lambda\alpha.\,\mathsf{new}\,\alpha{=}\alpha\ \mathsf{in}\ \delta_2(\mathrm{Wr}^-_{\tau'}(v_2\,\alpha))) \in V^{\circ,-}_n[\![\forall\alpha.\tau']\!]\rho$
– Suppose $(k'', w'') \sqsupset (k', w') \sqsupseteq (k, w)$ and $(\tau_1, \tau_2, r) \in T^{\circ,+}_{k'}[\![\Omega]\!]w'$.
– To show: $(k'', w'', \mathsf{new}\,\alpha{\approx}\tau_1\ \mathsf{in}\ \delta_1(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_1)\,\alpha)), \mathsf{new}\,\alpha{\approx}\tau_2\ \mathsf{in}\ \delta_2(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_2)\,\alpha))) \in E^{\circ,-}_n[\![\tau']\!]\rho, \alpha{\mapsto}r$
– So suppose $w''.\sigma_1; \mathsf{new}\,\alpha{\approx}\tau_1\ \mathsf{in}\ \delta_1(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_1)\,\alpha))$ terminates in $1{+}1{+}j_1{+}j_2 =: j < k$ steps:

$$\begin{array}{rl} & w''.\sigma_1; \mathsf{new}\,\alpha{\approx}\tau_1\ \mathsf{in}\ \delta_1(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_1)\,\alpha)) \\ \hookrightarrow^1 & w''.\sigma_1, \alpha_1{\approx}\tau_1; \delta'_1(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_1)\,\alpha_1)) \\ \hookrightarrow^1 & w''.\sigma_1, \alpha_1{\approx}\tau_1; \delta'_1(\mathrm{Wr}^-_{\tau'}(e_1[\alpha_1/\alpha])) \\ \hookrightarrow^{j_1} & \sigma'_1; \delta'_1(\mathrm{Wr}^-_{\tau'}(v'_1)) \\ \hookrightarrow^{j_2} & \sigma_1; v''_1 \end{array}$$

where $\delta'_1 := \delta_1, \alpha{\mapsto}\alpha_1$
– Note that

$$\begin{array}{rl} & w''.\sigma; \mathsf{new}\,\alpha{\approx}\tau_2\ \mathsf{in}\ \delta_2(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_2)\,\alpha)) \\ \hookrightarrow^1 & w''.\sigma, \alpha_2{\approx}\tau_2; \delta'_2(\mathrm{Wr}^-_{\tau'}((\Lambda\alpha.e_2)\,\alpha_2)) \\ \hookrightarrow^1 & w''.\sigma, \alpha_2{\approx}\tau_2; \delta'_2(\mathrm{Wr}^-_{\tau'}(e_2[\alpha_2/\alpha])) \end{array}$$

where $\delta'_2 := \delta_2, \alpha{\mapsto}\alpha_2$

- Let $w''_\alpha := \lfloor w'' \rfloor_{k''-1} \uplus (\alpha_1 \approx \tau_1, \alpha_2 \approx \tau_2, \alpha \mapsto (\alpha_1, \alpha_2), \alpha \mapsto \lfloor r \rfloor_{k''-1})$, so $(k''-1, w''_\alpha) \sqsupset (k'', w'')$.
- Let $r' := (w''_\alpha.\rho^1(\alpha), w''_\alpha.\rho^2(\alpha), V^{\bullet}_{k''-1}[\![\alpha]\!]w''_\alpha.\rho) = \lfloor r \rfloor_{k-1}$, so $(\alpha_1, \alpha_2, r') \in T^{-,\bullet}_{k''-1}[\![\Omega]\!]w''_\alpha$.
- Instantiating the assumption with $(k''-2, \lfloor w''_\alpha \rfloor_{k''-2}) \sqsupset (k''-1, w''_\alpha) \sqsupseteq (k'', w'')$ and (α_1, α_2, r') yields $(k''-2, \lfloor w''_\alpha \rfloor_{k''-2}, e_1[\alpha_1/\alpha], e_2[\alpha_2/\alpha]) \in E^{+,\bullet}_n[\![\tau']\!]\rho, \alpha \mapsto r'$.
- Consequently there exists $(k''-2-j_1, w''') \sqsupseteq (k''-2, \lfloor w''_\alpha \rfloor_{k''-2})$ such that $w''.\sigma_2, \alpha_2 \approx \tau_2; \delta'_2(\mathrm{Wr}^-_{\tau'}(e_2[\alpha_2/\alpha])) \hookrightarrow^* w'''.\sigma_2; \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2))$ with $w'''.\sigma_1 = \sigma'_1$ and $(k''-2-j_1, w''', v'_1, v'_2) \in V^{+,\bullet}_n[\![\tau']\!]\rho, \alpha \mapsto r'$.
- By Lemma 2.34, $(\alpha_1, \alpha_2, r') \in T^\circ_{k''-1}[\![\Omega]\!]w''_\alpha$.
- Furthermore $(\delta_1, \delta_2, \lfloor \rho \rfloor_{k''-2}) \in D^\circ_{k''-2}[\![\Delta]\!]w''_\alpha$ by Lemma 2.7, so $(\delta'_1, \delta'_2, (\lfloor \rho \rfloor_{k''-2}, \alpha \mapsto r')) \in D^\circ_{k''-2}[\![\Delta, \alpha]\!]w''_\alpha$.
- Hence by induction, $(k''-2-j_1, w''', \delta'_1(\mathrm{Wr}^-_{\tau'}(v'_1)), \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2))) \in E^{\circ,-}_n[\![\tau']\!]\lfloor \rho \rfloor_{k''-2}, \alpha \mapsto r'$.
- Consequently, there exists $(k''-j, w'''') \sqsupseteq (k''-2-j_1, w''')$ such that $w'''.\sigma_2; \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2)) \hookrightarrow^* w''''.\sigma_2; v''_2$ such that $w''''.\sigma_1 = \sigma_1$ and $(k''-j, w'''', v''_1, v''_2) \in V^{\circ,-}_n[\![\tau']\!]\lfloor \rho \rfloor_{k''-2}, \alpha \mapsto r'$.
- By Restriction (2.4) and Irrelevance (2.6), $(k''-j, w'''', v''_1, v''_2) \in V^{\circ,-}_n[\![\tau']\!]\rho, \alpha \mapsto r$.

- Case $\tau = \exists \alpha.\tau'$: $v_i = \mathsf{pack}\langle \tau_i, v'_i \rangle$
  - To show: $(k, w, \mathsf{unpack}\langle \alpha, x \rangle{=}v_1 \mathsf{\ in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^-_{\tau'}(x)) \rangle$, $\mathsf{unpack}\langle \alpha, x \rangle{=}v_2 \mathsf{\ in\ pack}\langle \alpha, \delta_2(\mathrm{Wr}^-_{\tau'}(x)) \rangle) \in E^{\circ,-}_n[\![\exists \alpha.\tau']\!]\rho$
  - So suppose $w.\sigma_1; \mathsf{unpack}\langle \alpha, x \rangle{=}v_1 \mathsf{\ in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^-_{\tau'}(x)) \rangle$ terminates in $1+j' =: j < k$ steps:

$$\begin{array}{ll} & w.\sigma_1; \mathsf{unpack}\langle \alpha, x \rangle{=}v_1 \mathsf{\ in\ pack}\langle \alpha, \delta_1(\mathrm{Wr}^-_{\tau'}(x)) \rangle \\ \hookrightarrow^1 & w.\sigma_1; \mathsf{pack}\langle \tau_1, \delta'_1(\mathrm{Wr}^-_{\tau'}(v'_1)) \rangle \\ \hookrightarrow^{j'} & \sigma_1; \mathsf{pack}\langle \tau_1, v''_1 \rangle \end{array}$$

    where $\delta'_1 := \delta_1, \alpha \mapsto \tau_1$
  - Note that

$$\begin{array}{ll} & w.\sigma_2; \mathsf{unpack}\langle \alpha, x \rangle{=}v_2 \mathsf{\ in\ pack}\langle \alpha, \delta_2(\mathrm{Wr}^-_{\tau'}(x)) \rangle \\ \hookrightarrow^1 & w.\sigma_2; \mathsf{pack}\langle \tau_2, \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2)) \rangle \end{array}$$

    where $\delta'_2 := \delta_2, \alpha \mapsto \tau_2$
  - By assumption we know $(k-1, \lfloor w \rfloor_{k-1}, v'_1, v'_2) \in V^{+,\bullet}_n[\![\tau']\!]\rho, \alpha \mapsto r$ for some r with $(\tau_1, \tau_2, r) \in T^{+,\bullet}_k[\![\Omega]\!]w$.
  - By Lemma 2.34, $(\tau_1, \tau_2, r) \in T^\circ_k[\![\Omega]\!]w$.
  - Furthermore $(\delta_1, \delta_2, \lfloor \rho \rfloor_k) \in D^\circ_k[\![\Delta]\!]w$ by Lemma 2.7, so $(\delta'_1, \delta'_2, (\lfloor \rho \rfloor_k, \alpha \mapsto r)) \in D^\circ_k[\![\Delta, \alpha]\!]w$.
  - Hence induction yields $(k-1, \lfloor w \rfloor_{k-1}, \delta'_1(\mathrm{Wr}^-_{\tau'}(v'_1)), \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2))) \in E^{\circ,-}_n[\![\tau']\!]\lfloor \rho \rfloor_k, \alpha \mapsto r$.
  - Consequently, there exists $(k-j, w') \sqsupseteq (k-1, \lfloor w \rfloor_{k-1})$ such that $w.\sigma_2; \mathsf{pack}\langle \tau_2, \delta'_2(\mathrm{Wr}^-_{\tau'}(v'_2)) \rangle \hookrightarrow^* w'.\sigma_2; \mathsf{pack}\langle \tau_2, v''_2 \rangle$ with $w'.\sigma_1 = \sigma_1$ and $(k-j, w', v''_1, v''_2) \in V^{\circ,-}_n[\![\tau']\!]\lfloor \rho \rfloor_k, \alpha \mapsto r$.

- By Restriction (2.4) and Irrelevance (2.6), $(k-j, w', v_1'', v_2'') \in V_n^{\circ,-}[\![\tau']\!]\rho, \alpha \mapsto r$.
- Since $(\tau_1, \tau_2, r) \in T_k^{\circ,-}[\![\Omega]\!]w$ by Lemma 2.34, $(k-j, w', \mathsf{pack}\langle\tau_1, v_1''\rangle, \mathsf{pack}\langle\tau_2, v_2''\rangle) \in V_n^{\circ,-}[\![\exists\alpha.\tau']\!]\rho$ follows by Closure Under World Extension (2.8).

(b) Symmetric to (1b).

■

Theorem 2.37 (Wrapping II)
Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^{\circ}[\![\Delta]\!]w_0$, $(k, w) \sqsupset (n, w_0)$, and $\Delta \vdash \tau$.

1. (a) If $(k, w, v_1, v_2) \in V_n^{\circ}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^+(v_1)), \delta_2(\mathrm{Wr}_\tau^+(v_2))) \in E_n^{\bullet}[\![\tau]\!]\rho$.
   (b) If $(k, w, e_1, e_2) \in E_n^{\circ}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^+(e_1)), \delta_2(\mathrm{Wr}_\tau^+(e_2))) \in E_n^{\bullet}[\![\tau]\!]\rho$.
2. (a) If $(k, w, v_1, v_2) \in V_n^{\bullet}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^-(v_1)), \delta_2(\mathrm{Wr}_\tau^-(v_2))) \in E_n^{\circ}[\![\tau]\!]\rho$.
   (b) If $(k, w, e_1, e_2) \in E_n^{\bullet}[\![\tau]\!]\rho$, then $(k, w, \delta_1(\mathrm{Wr}_\tau^-(e_1)), \delta_2(\mathrm{Wr}_\tau^-(e_2))) \in E_n^{\circ}[\![\tau]\!]\rho$.

Proof: Follows immediately from Wrapping I (2.36) and Chain (2.35). ■

Corollary 2.38
If $\vdash e_1 \precsim^{\circ} e_2 : \tau$, then $\vdash \mathrm{Wr}^+(e_1) \precsim^{\bullet} \mathrm{Wr}^+(e_2) : \tau$.

3 Examples

3.1 Semaphore ADT, ∃-version

$$\begin{aligned}
\tau &:= \alpha \times (\alpha \to \alpha) \times (\alpha \to \mathsf{bool}) \\
v_1 &:= \langle \mathsf{true}, \lambda x{:}\alpha.\neg x, \lambda x{:}\alpha.x \rangle \\
v_2 &:= \langle 1, \lambda x{:}\alpha.1 - x, \lambda x{:}\alpha.x \neq 0 \rangle \\
e_1 &:= \mathsf{new}\ \alpha{\approx}\mathsf{bool}\ \mathsf{in}\ \mathsf{pack}\langle \alpha, v_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau \\
e_2 &:= \mathsf{new}\ \alpha{\approx}\mathsf{int}\ \mathsf{in}\ \mathsf{pack}\langle \alpha, v_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau
\end{aligned}$$

We show $\epsilon; \epsilon \vdash e_1 \precsim^{\bullet} e_2 : \exists\alpha.\tau$; the other direction is proven analogously.

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^{\bullet}[\![\epsilon]\!]w_0$, and $(k, w, \gamma_1, \gamma_2) \in G_n^{\bullet}[\![\epsilon]\!]\rho$ with $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(e_1), \delta_2\gamma_2(e_2)) \in E_n^{\bullet}[\![\exists\alpha.\tau]\!]\rho$, i.e., $(k, w, e_1, e_2) \in E_n^{\bullet}[\![\exists\alpha.\tau]\!]\rho$
- We know:
  - $w.\sigma_1; e_1 \hookrightarrow^1 w.\sigma_1, \alpha_1{\approx}\mathsf{bool}; \mathsf{pack}\langle \alpha_1, v_1[\alpha_1/\alpha] \rangle\ \mathsf{as}\ \exists\alpha.\tau$
  - $w.\sigma_2; e_2 \hookrightarrow^* w.\sigma_1, \alpha_2{\approx}\mathsf{int}; \mathsf{pack}\langle \alpha_2, v_2[\alpha_2/\alpha] \rangle\ \mathsf{as}\ \exists\alpha.\tau$

- Let

$$R := \{(k', w', v_a, v_b) \in \text{Atom}_{k-1} \mid (v_a, v_b) = (\mathsf{true}, 1) \vee (v_a, v_b) = (\mathsf{false}, 0)\}$$
$$w_\alpha := \lfloor w \rfloor_{k-1} \uplus (\alpha_1 {\approx} \mathsf{bool}, \alpha_2 {\approx} \mathsf{int}, \alpha {\mapsto} (\alpha_1, \alpha_2), \alpha {\mapsto} R)$$

  such that $(k-1, w_\alpha) \sqsupset (k, w)$.

- We claim $(k-1, w_\alpha, \mathsf{pack}\langle \alpha_1, v_1[\alpha_1/\alpha]\rangle \text{ as } \exists\alpha.\tau, \mathsf{pack}\langle \alpha_2, v_2[\alpha_2/\alpha]\rangle \text{ as } \exists\alpha.\tau) \in V_n^\bullet[\![\exists\alpha.\tau]\!]\rho$.
- Note that $\alpha_1 = w_\alpha^1(\alpha)$ and $\alpha_2 = w_\alpha^2(\alpha)$.
- So suppose $(k', w') \sqsupset (k-1, w_\alpha)$.
- To show: $(k', w', v_1[\alpha_1/\alpha], v_2[\alpha_2/\alpha]) \in V_n^\bullet[\![\alpha \times (\alpha \to \alpha) \times (\alpha \to \mathsf{bool})]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$, which decomposes into three parts:
  1. $(k', w', \mathsf{true}, 1) \in V_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$, which holds since $V_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho = V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho \cap \text{Atom}_n = w_\alpha.\rho(\alpha).R \cap \text{Atom}_{k-1} = R \ni (k', w', \mathsf{true}, 1)$.
  2. $(k', w', \lambda x{:}\alpha_1.\neg x, \lambda x{:}\alpha_2.1 - x) \in V_n^\bullet[\![\alpha \to \alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$
     - Suppose $(k'', w'', v_3, v_4) \in V_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$ for $(k'', w'') \sqsupseteq (k', w')$, i.e., $(k'', w'', v_3, v_4) \in R$, so either $(v_3, v_4) = (\mathsf{true}, 1)$ or $(v_3, v_4) = (\mathsf{false}, 0)$.
     - To show: $(k'', w'', \neg v_3, 1 - v_4) \in E_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$.
     - If $(v_3, v_4) = (\mathsf{true}, 1)$, then $w''.\sigma_1; \neg v_3 \hookrightarrow^1 w''.\sigma_1; \mathsf{false}$ and $w''.\sigma_2; 1 - v_4 \hookrightarrow^* w''.\sigma_2; 0$.
     - Note that $(k''-1, \lfloor w'' \rfloor_{k''-1}, \mathsf{false}, 0) \in R = V_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$.
     - Similarly for $(v_3, v_4) = (\mathsf{false}, 0)$.
  3. $(k', w', \lambda x{:}\alpha_1.x, \lambda x{:}\alpha_2.x \neq 0) \in V_n^\bullet[\![\alpha \to \mathsf{bool}]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$
     - Suppose $(k'', w'', v_3, v_4) \in V_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$ for $(k'', w'') \sqsupseteq (k', w')$, i.e., $(k'', w'', v_3, v_4) \in R$, so either $(v_3, v_4) = (\mathsf{true}, 1)$ or $(v_3, v_4) = (\mathsf{false}, 0)$.
     - To show: $(k'', w'', v_3, v_4 \neq 0) \in E_n^\bullet[\![\alpha]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$
     - If $(v_3, v_4) = (\mathsf{true}, 1)$, then $w''.\sigma_1; v_3 \hookrightarrow^0 w''.\sigma_1; \mathsf{true}$ and $w''.\sigma_2; v_4 \neq 0 \hookrightarrow^* w''.\sigma_2; \mathsf{true}$.
     - Note that $(k''-1, \lfloor w'' \rfloor_{k''-1}, \mathsf{true}, \mathsf{true}) \in V_n^\bullet[\![\mathsf{bool}]\!]\rho, \alpha {\mapsto} V_{k-1}^\bullet[\![\alpha]\!]w_\alpha.\rho$.
     - Similarly for $(v_3, v_4) = (\mathsf{false}, 0)$.

3.2 Semaphore ADT, ∀-version

$$\tau := \alpha \times (\alpha \to \alpha) \times (\alpha \to \mathsf{bool})$$
$$v_1 := \langle \mathsf{true}, \lambda x{:}\mathsf{bool}.\neg x, \lambda x{:}\mathsf{bool}.x\rangle$$
$$v_2 := \langle 1, \lambda x{:}\mathsf{int}.x - 1, \lambda x{:}\mathsf{int}.x \neq 0\rangle$$
$$e_1 := \lambda x{:}\forall\alpha.\tau \to \mathsf{bool}.(\mathsf{new}\, \alpha {\approx} \mathsf{bool} \text{ in } x\, \alpha)\, v_1$$
$$e_2 := \lambda x{:}\forall\alpha.\tau \to \mathsf{bool}.(\mathsf{new}\, \alpha {\approx} \mathsf{int} \text{ in } x\, \alpha)\, v_2$$

We show $\epsilon; \epsilon \vdash e_1 \precsim^\bullet e_2 : (\forall\alpha.\tau \to \mathsf{bool}) \to \mathsf{bool}$; the other direction is proven analogously.

- Suppose $w_0 \in \text{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\epsilon]\!]w_0$, and $(k, w, \gamma_1, \gamma_2) \in G_n^\bullet[\![\epsilon]\!]\rho$ with $(k, w) \sqsupset (n, w_0)$.

- To show: $(k, w, w.\eta^1\delta\gamma_1(e_1), w.\eta^2\delta\gamma_2(e_2)) \in E_n^\bullet[\![(\forall\alpha.\tau \to \mathsf{bool}) \to \mathsf{bool}]\!]\rho$, i.e., $(k, w, e_1, e_2) \in V_n^\bullet[\![(\forall\alpha.\tau \to \mathsf{bool}) \to \mathsf{bool}]\!]\rho$.
- So suppose $(k', w', \Lambda\alpha.e_3, \Lambda\alpha.e_4) \in V_n^\bullet[\![\forall\alpha.\tau \to \mathsf{bool}]\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.
- To show: $(k', w', (\mathsf{new}\,\alpha{\approx}\mathsf{bool}\ \mathsf{in}\ (\Lambda\alpha.e_3)\,\alpha)\,v_1, (\mathsf{new}\,\alpha{\approx}\mathsf{int}\ \mathsf{in}\ (\Lambda\alpha.e_4)\,\alpha)\,v_2) \in E_n^\bullet[\![\mathsf{bool}]\!]\rho$.
- Suppose $w'.\sigma_1; (\mathsf{new}\,\alpha{\approx}\mathsf{bool}\ \mathsf{in}\ (\Lambda\alpha.e_3)\,\alpha)\,v_1$ terminates in $2 + j_1 + 1 + j_2 =: j < k'$ steps:

$$\begin{array}{ll} & w'.\sigma_1; (\mathsf{new}\,\alpha{\approx}\mathsf{bool}\ \mathsf{in}\ (\Lambda\alpha.e_3)\,\alpha)\,v_1 \\ \hookrightarrow^2 & w'.\sigma_1, \alpha_1{\approx}\mathsf{bool}; e_3[\alpha_1/\alpha]\,v_1 \\ \hookrightarrow^{j_1} & \sigma_1'; (\lambda x.e_3')\,v_1 \\ \hookrightarrow^1 & \sigma_1'; e_3'[v_1/x] \\ \hookrightarrow^{j_2} & \sigma_1; v_3 \end{array}$$

- We need to show the existence of $(k' - j, w'''') \sqsupset (k', w')$ such that $w'.\sigma_2; (\mathsf{new}\,\alpha{\approx}\mathsf{bool}\ \mathsf{in}\ (\Lambda\alpha.e_4)\,\alpha)\,v_2 \hookrightarrow^* w''''.\sigma_2; v_4$ with $w''''.\sigma_1 = \sigma_1$ and $(k' - j, w'''', v_3, v_4) \in V_n^\bullet[\![\mathsf{bool}]\!]\rho$.
- Let

$$\begin{aligned} R &:= \{(k'', w'', v_a, v_b) \in \mathrm{Atom}_{k'-1} \mid (v_a, v_b) = (\mathsf{true}, 1) \vee (v_a, v_b) = (\mathsf{false}, 0)\} \\ w'_\alpha &:= \lfloor w' \rfloor_{k'-1} \uplus (\alpha_1{\approx}\mathsf{bool}, \alpha_2{\approx}\mathsf{int}, \alpha{\mapsto}(\alpha_1, \alpha_2), \alpha{\mapsto}R) \\ w'' &:= \lfloor w'_\alpha \rfloor_{k'-2} \end{aligned}$$

- Since $(k' - 2, w'') \sqsupset (k' - 1, w'_\alpha) \sqsupset (k', w')$ and $\alpha \in \mathrm{Typ}_{\mathrm{dom}(w'_\alpha.\rho)}$, we know $(k' - 2, w'', e_3[\alpha_1/\alpha], e_4[\alpha_2/\alpha]) \in E_n^\bullet[\![\tau \to \mathsf{bool}]\!]\rho, \alpha{\mapsto}V_{k'-1}^\bullet[\![\alpha]\!]w'_\alpha.\rho$.
- Consequently, there exists $(k' - 2 - j_1, w''') \sqsupseteq (k' - 2, w'')$ such that $w'.\sigma_2, \alpha_2{\approx}\mathsf{int}; e_4[\alpha_2/\alpha]\,v_2 \hookrightarrow^* w'''.\sigma_2; (\lambda x.e_4')\,v_2$ with $w'''.\sigma_1 = \sigma_1'$ and $(k' - 2 - j_1, w''', \lambda x.e_3', \lambda x.e_4') \in V_n^\bullet[\![\tau \to \mathsf{bool}]\!]\rho, \alpha{\mapsto}V_{k'-1}^\bullet[\![\alpha]\!]w'_\alpha.\rho$.
- If we can show $(k' - 2 - j_1 - 1, \lfloor w''' \rfloor_{k'-2-j_1-1}, v_1, v_2) \in V_n^\bullet[\![\tau]\!]\rho, \alpha{\mapsto}V_{k'-1}^\bullet[\![\alpha]\!]w'_\alpha.\rho$, then the latter yields $(k' - 2 - j_1 - 1, \lfloor w''' \rfloor_{k'-2-j_1-1}, e_3'[v_1/x], e_4'[v_2/x]) \in E_n^\bullet[\![\mathsf{bool}]\!]\rho, \alpha{\mapsto}V_{k'-1}^\bullet[\![\alpha]\!]w'_\alpha.\rho$, which in turns implies the existence of the wanted $(k' - j, w'''')$.
- Showing $(k' - 2 - j_1 - 1, \lfloor w''' \rfloor_{k'-2-j_1-1}, v_1, v_2) \in V_n^\bullet[\![\tau]\!]\rho, \alpha{\mapsto}V_{k'-1}^\bullet[\![\alpha]\!]w'_\alpha.\rho$ decomposes and is shown as in the previous example.

3.3 Double Effect

$$\begin{aligned} v_1 &:= \lambda x{:}\,\mathsf{unit} \to \tau.(\lambda\_{:}b.\,x\,c)\,(x\,c) \\ v_2 &:= \lambda x{:}\,\mathsf{unit} \to \tau.x\,c \end{aligned}$$

We show $\epsilon; \epsilon \vdash v_1 \precsim^\bullet v_2 : (\mathsf{unit} \to \tau) \to \tau$:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\bullet[\![\Gamma]\!]\rho$, and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(v_1), \delta_2\gamma_2(v_2)) \in E_n^\bullet[\![(\mathsf{unit} \to \tau) \to \tau]\!]\rho$, i.e., $(k, w, v_1, v_2) \in V_n^\bullet[\![(\mathsf{unit} \to \tau) \to \tau]\!]\rho$

- So suppose $(k', w', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.
- To show: $(k', w', (\lambda\_{:}\tau.(\lambda x.e_1)\,c)\,((\lambda x.e_1)\,c), (\lambda x.e_2)\,c) \in E_n^\bullet[\![\tau]\!]\rho$
- Suppose $w'.\sigma_1; (\lambda\_{:}\tau.(\lambda x.e_1)\,c)\,((\lambda x.e_1)\,c)$ terminates in $3 + 2j' =: j < k'$ steps:

$$\begin{array}{ll} & w'.\sigma_1; (\lambda\_.(\lambda x.e_1)\,c)\,((\lambda x.e_1)\,c) \\ \hookrightarrow^1 & w'.\sigma_1; (\lambda\_.(\lambda x.e_1)\,c)\,e_1[c/x] \\ \hookrightarrow^{j'} & \sigma_1'; (\lambda\_.(\lambda x.e_1)\,c)\,v_1' \\ \hookrightarrow^1 & \sigma_1'; (\lambda x.e_1)\,c \\ \hookrightarrow^1 & \sigma_1'; e_1[c/x] \\ \hookrightarrow^{j'} & \sigma_1; v_1'' \end{array}$$

- Let $w_1' := (\sigma_1', w'.\sigma_2, w'.\eta, w'.\rho)$, so $(k', w_1') \sqsupseteq (k', w')$.
- Since we have $(k' - j' - 2, \lfloor w_1' \rfloor_{k'-j'-2}, \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ by Closure Under World Extension and $(k' - j' - 3, \lfloor w_1' \rfloor_{k'-j'-3}, c, c) \in V_n^\bullet[\![\mathsf{unit}]\!]\rho$, we get $(k' - j' - 3, \lfloor w_1' \rfloor_{k'-j'-3}, e_1[c/x], e_2[c/x]) \in E_n^\bullet[\![\tau]\!]\rho$.
- Consequently, there exists $(k' - j, w'') \sqsupseteq (k' - j' - 3, \lfloor w_1' \rfloor_{k'-j'-3})$ such that $w'.\sigma_2; e_2[c/x] \hookrightarrow^* w''.\sigma_2; v_2'$ with $w''.\sigma_1 = \sigma_1$ and $(k' - j, w'', v_1'', v_2') \in V_n^\bullet[\![\tau]\!]\rho$.

We show $\epsilon; \epsilon \vdash v_2 \precsim^\bullet v_1 : (\mathsf{unit} \to \tau) \to \tau$:

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\bullet[\![\Gamma]\!]\rho$, and $(k, w) \sqsupset (n, w_0)$.
- To show: $(k, w, \delta_1\gamma_1(v_2), \delta_2\gamma_2(v_1)) \in E_n^\bullet[\![(\mathsf{unit} \to \tau) \to \tau]\!]\rho$, i.e., $(k, w, v_2, v_1) \in V_n^\bullet[\![(\mathsf{unit} \to \tau) \to \tau]\!]\rho$
- So suppose $(k', w', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.
- To show: $(k', w', (\lambda x.e_1)\,c, (\lambda\_{:}\tau.(\lambda x.e_2)\,c)\,((\lambda x.e_2)\,c)) \in E_n^\bullet[\![\tau]\!]\rho$
- Suppose $w'.\sigma_1; (\lambda x.e_1)\,c$ terminates in $1 + j' =: j < k'$ steps:

$$\begin{array}{ll} & w'.\sigma_1; (\lambda x.e_1)\,c \\ \hookrightarrow^1 & w'.\sigma_1; e_1[c/x] \\ \hookrightarrow^{j'} & \sigma_1; v_1' \end{array}$$

- Instantiating $(k', w', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ with $(k' - 1, \lfloor w' \rfloor_{k'-1}, c, c) \in V_n^\bullet[\![\mathsf{unit}]\!]\rho$ yields $(k' - 1, \lfloor w' \rfloor_{k'-1}, e_1[c/x], e_2[c/x]) \in E_n^\bullet[\![\tau]\!]\rho$.
- Consequently there exists $(k' - j, w'') \sqsupseteq (k' - 1, \lfloor w' \rfloor_{k'-1})$ such that $w'.\sigma_2; (\lambda\_.(\lambda x.e_2)\,c)\,e_2[c/x] \hookrightarrow^* w''.\sigma_2; (\lambda\_.(\lambda x.e_2)\,c)\,v_2'$.
- Let $w_2' = (w'.\sigma_1, w''.\sigma_2, w'.\eta, w'.\rho)$, so $(k', w_2') \sqsupseteq (k', w')$.
- By Closure Under World Extension (2.8) we have $(k', w_2', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$.
- Instantiating this with $(k' - 1, \lfloor w_2' \rfloor_{k'-1}, c, c) \in V_n^\bullet[\![\mathsf{unit}]\!]\rho$ yields $(k' - 1, \lfloor w_2' \rfloor_{k'-1}, e_1[c/x], e_2[c/x]) \in E_n^\bullet[\![\tau]\!]\rho$.

- Consequently there exists $(k'-j, w''') \sqsupseteq (k'-1, \lfloor w_2' \rfloor_{k'-1})$ such that $w''.\sigma_2; e_2[c/x] \hookrightarrow^* w'''.\sigma_2; v_2''$ with $w'''.\sigma_1 = \sigma_1$ and $(k'-j, w''', v_1', v_2'') \in V_n^\bullet[\![\tau]\!]\rho$.

- Note that

$$\begin{array}{ll} & w'.\sigma_2; (\lambda\_.(\lambda x.e_2)\,c)\,((\lambda x.e_2)\,c) \\ \hookrightarrow^1 & w'.\sigma_2; (\lambda\_.(\lambda x.e_2)\,c)\,e_2[c/x] \\ \hookrightarrow^* & w''.\sigma_2; (\lambda\_.(\lambda x.e_1)\,c)\,v_2' \\ \hookrightarrow^1 & w''.\sigma_2; (\lambda x.e_2)\,c \\ \hookrightarrow^1 & w''.\sigma_2; e_2[c/x] \\ \hookrightarrow^* & w'''.\sigma_2; v_2' \end{array}$$

3.4 Effect Order

$$\begin{aligned} v_1 &:= \lambda x_f\colon \mathsf{unit} \to \tau.\lambda x_g\colon \mathsf{unit} \to \tau'.\ \mathsf{let}\, x_g' = x_g\,()\ \mathsf{in}\ \langle x_f\,(), x_g' \rangle \\ v_2 &:= \lambda x_f\colon \mathsf{unit} \to \tau.\lambda x_g\colon \mathsf{unit} \to \tau'.\ \langle x_f\,(), x_g\,() \rangle \end{aligned}$$

We show $\epsilon; \epsilon \vdash v_1 \precsim v_2 : (\mathsf{unit} \to \tau) \to (\mathsf{unit} \to \tau') \to (\tau \times \tau')$.

- Suppose $w_0 \in \mathrm{World}_n$, $(\delta_1, \delta_2, \rho) \in D_n^\bullet[\![\Delta]\!]w_0$, $(k, w, \gamma_1, \gamma_2) \in G_n^\bullet[\![\Gamma]\!]\rho$, and $(k, w) \sqsupset (n, w_0)$.

- To show: $(k, w, \delta_1\gamma_1(v_1), \delta_2\gamma_2(v_2)) \in E_n^\bullet[\![(\mathsf{unit} \to \tau) \to (\mathsf{unit} \to \tau') \to (\tau \times \tau')]\!]\rho$, i.e., $(k, w, v_1, v_2) \in V_n^\bullet[\![(\mathsf{unit} \to \tau) \to (\mathsf{unit} \to \tau') \to (\tau \times \tau')]\!]\rho$

- So suppose $(k', w', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ where $(k', w') \sqsupseteq (k, w)$.

- To show: $(k', w', \lambda x_g\colon \mathsf{unit} \to \tau'.\ \mathsf{let}\, x_g' = x_g\,()\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle,$
  $\lambda x_g\colon \mathsf{unit} \to \tau'.\ \langle (\lambda x.e_1)\,(), x_g\,() \rangle) \in V_n^\bullet[\![(\mathsf{unit} \to \tau') \to (\tau \times \tau')]\!]\rho$

- So suppose $(k'', w'', \lambda x.e_3, \lambda x.e_4) \in V_n^\bullet[\![\mathsf{unit} \to \tau']\!]\rho$ where $(k'', w'') \sqsupseteq (k', w')$.

- To show: $(k'', w'', \mathsf{let}\, x_g' = (\lambda x.e_3)\,()\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle, \langle (\lambda x.e_1)\,(), (\lambda x.e_4)\,() \rangle) \in V_n^\bullet[\![\tau \times \tau']\!]\rho$

- Suppose $w''.\sigma_1; \mathsf{let}\, x_g' = (\lambda x.e_3)\,()\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle$ terminates in $1 + j_1 + 1 + 1 + j_2 =: j < k'$ steps:

$$\begin{array}{ll} & w''.\sigma_1; \mathsf{let}\, x_g' = (\lambda x.e_3)\,()\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle \\ \hookrightarrow^1 & w''.\sigma_1; \mathsf{let}\, x_g' = e_3[()/x]\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle \\ \hookrightarrow^{j_1} & \sigma_1'; \mathsf{let}\, x_g' = v_3'\ \mathsf{in}\ \langle (\lambda x.e_1)\,(), x_g' \rangle \\ \hookrightarrow^1 & \sigma_1'; \langle (\lambda x.e_1)\,(), v_3' \rangle \\ \hookrightarrow^1 & \sigma_1'; \langle e_1[()/x], v_3' \rangle \\ \hookrightarrow^{j_2} & \sigma_1; \langle v_1', v_3' \rangle \end{array}$$

- Let $(k_2', w_2') := (k'' - 1 - j_1 - 1 - 1, (\sigma_1', w''.\sigma_2, w''.\eta, \lfloor w''.\rho \rfloor_{k''-1j_1-1-1}))$, so $(k_2', w_2') \sqsupseteq (k'', w'')$.

- Since we have $(k_2', w_2', \lambda x.e_1, \lambda x.e_2) \in V_n^\bullet[\![\mathsf{unit} \to \tau]\!]\rho$ by Closure Under World Extension and $(k_2' - 1, \lfloor w_2' \rfloor_{k_2'-1}, (), ()) \in V_n^\bullet[\![\mathsf{unit}]\!]\rho$, we get $(k_2' - 1, \lfloor w_2' \rfloor_{k_2'-1}, e_1[()/x], e_2[()/x]) \in E_n^\bullet[\![\tau]\!]\rho$.

- Note that $w_2'.\sigma_1 = \sigma_1'$.

- Consequently, there exists $(k'' - j, w_2'') \sqsupseteq (k_2' - 1, \lfloor w_2' \rfloor_{k_2'-1})$ such that
  $w''.\sigma_2; e_2[()/x] \hookrightarrow^* w_2''.\sigma_2; v_2'$ with $w_2''.\sigma_1 = \sigma_1$ and $(k'' - j, w_2'', v_1', v_2') \in V_n^\bullet[\![\tau]\!]\rho$.

- Let $(k'_1, w'_1) := (k'' - 1, (w''.\sigma_1, w''_2.\sigma_2, w''.\eta, \lfloor w''.\rho \rfloor_{k''-1}))$, so $(k'_1, w'_1) \sqsupseteq (k'', w'')$.
- Since we have $(k'_1, w'_1, \lambda x.e_3, \lambda x.e_4) \in V^\bullet_n[\![\mathsf{unit} \to \tau']\!]\rho$ by Closure Under World Extension and $(k'_1 - 1, \lfloor w'_1 \rfloor_{k'_1-1}, (), ()) \in V^\bullet_n[\![\mathsf{unit}]\!]\rho$, we get $(k'_1 - 1, \lfloor w'_1 \rfloor_{k'_2-1}, e_3[()/x], e_4[()/x]) \in E^\bullet_n[\![\tau']\!]\rho$.
- Note that $w'_1.\sigma_1 = w''.\sigma_1$.
- Consequently, there exists $(k'' - 1 - j_1, w''_1) \sqsupseteq (k'_1 - 1, \lfloor w'_1 \rfloor_{k'_1-1})$ such that $w''_2.\sigma_2; e_4[()/x] \hookrightarrow^* w''_1.\sigma_2; v'_4$ with $w''_1.\sigma_1 = \sigma'_1$ and $(k'' - 1 - j_1, w''_1, v'_3, v'_4) \in V^\bullet_n[\![\tau']\!]\rho$.
- W.l.o.g. $(\mathrm{dom}(w''_1.\eta) \setminus \mathrm{dom}(w'_1.\eta)) \cap (\mathrm{dom}(w''_2.\eta) \setminus \mathrm{dom}(w'_2.\eta)) = \emptyset$, so $w''_1.\eta \cup w''_2.\eta$ and $w''_1.\rho \cup w''_1.\rho$ are well-defined.
- Let $w_3 := (w''_2.\sigma_1, w''_1.\sigma_2, w''_1.\eta \cup w''_2.\eta, w''_1.\rho \cup w''_2.\rho)$.
- It remains to show the injectivity of $w_3.\eta^i$.
- Note that $\mathrm{rng}(w''_1.\eta^i) \setminus \mathrm{rng}(w'_1.\eta^i) \subseteq \mathrm{dom}(w''_1.\sigma_i) \setminus \mathrm{dom}(w'_1.\sigma_i)$ by definition of world extension.
- Similarly, $\mathrm{rng}(w''_2.\eta^i) \setminus \mathrm{rng}(w'_2.\eta^i) \subseteq \mathrm{dom}(w''_2.\sigma_i) \setminus \mathrm{dom}(w'_2.\sigma_i)$ by definition of world extension.
- Suppose $\alpha, \alpha' \in \mathrm{dom}(w_3.\eta)$.
- Case $\alpha, \alpha' \in \mathrm{dom}(w''.\eta)$: Trivial.
- Case $\alpha \in \mathrm{dom}(w''.\eta)$ and $\alpha' \in \mathrm{dom}(w''_1.\eta) \setminus \mathrm{dom}(w''.\eta)$:
  - Then we know $w_3.\eta^i(\alpha) \in \mathrm{dom}(w''.\sigma_i)$ and $w_3.\eta^i(\alpha') \in \mathrm{dom}(w''_1.\sigma_i) \setminus \mathrm{dom}(w'_1.\sigma_i)$.
  - Since $w'_1.\sigma_i \supseteq w''.\sigma_i$, we have $w_3.\eta^i(\alpha) \neq w_3.\eta^i(\alpha')$.
- Case $\alpha \in \mathrm{dom}(w''_1.\eta) \setminus \mathrm{dom}(w''.\eta)$ and $\alpha' \in \mathrm{dom}(w''_2.\eta) \setminus \mathrm{dom}(w''.\eta)$:
  - Then we know $w_3.\eta^i(\alpha) \in \mathrm{dom}(w''_1.\sigma_i) \setminus \mathrm{dom}(w'_1.\sigma_i)$ and $w_3.\eta^i(\alpha') \in \mathrm{dom}(w''_2.\sigma_i) \setminus \mathrm{dom}(w'_2.\sigma_i)$.
  - For $i = 1$ this means $w_3.\eta^1(\alpha) \in \mathrm{dom}(w''_1.\sigma_1) = \mathrm{dom}(\sigma'_1) = \mathrm{dom}(w'_2.\sigma_1)$, so it cannot equal $w_3.\eta^1(\alpha')$.
  - For $i = 2$ this means $w_3.\eta^2(\alpha) \in \mathrm{dom}(w''_1.\sigma_2) \setminus \mathrm{dom}(w''_2.\sigma_2)$, so it cannot equal $w_3.\eta^2(\alpha')$.

3.5 A Free Theorem

Suppose $\sigma_0 \vdash v : \forall\alpha.\alpha \to \alpha$. We want to show that either

1. $\sigma; \mathrm{Wr}^-_{\forall\alpha.\alpha\to\alpha}(v)\, \tau\, v' \uparrow$ for all σ, τ, v' with $\sigma \supseteq \sigma_0$ and $\sigma \vdash v' : \tau$, or
2. $\sigma; \mathrm{Wr}^-_{\forall\alpha.\alpha\to\alpha}(v)\, \tau\, v' \hookrightarrow^* \sigma'; v'$ for all σ, τ, v' with $\sigma \supseteq \sigma_0$ and $\sigma \vdash v' : \tau$, where σ' may be different each time.

In order to do that, we first show for all σ, τ, v' with $\sigma \supseteq \sigma_0$ and $\sigma \vdash v' : \tau$ that either

1. $\sigma; \mathrm{Wr}^-_{\forall\alpha.\alpha\to\alpha}(v)\, \tau\, v' \uparrow$, or
2. $\sigma; \mathrm{Wr}^-_{\forall\alpha.\alpha\to\alpha}(v)\, \tau\, v' \hookrightarrow^* \sigma'; v'$.

- If $\sigma; \mathrm{Wr}^-(v)\,\tau\,v' \uparrow$, there is nothing to show.
- So suppose $\sigma; \mathrm{Wr}^-(v)\,\tau\,v'$ terminates in $j := j_1 + 1 + j_2 + 1 + j_3$ steps:

$$\begin{array}{ll} & \sigma; \mathrm{Wr}^-(v)\,\tau\,v' \\ \hookrightarrow^{j_1} & \sigma_1; (\Lambda\alpha.e_1)\,\tau\,v' \\ \hookrightarrow^{1} & \sigma_1; e_1[\tau/\alpha]\,v' \\ \hookrightarrow^{j_2} & \sigma_2; (\lambda x.e_1')\,v' \\ \hookrightarrow^{1} & \sigma_2; e_1'[v'/x] \\ \hookrightarrow^{j_3} & \sigma_3; v_1 \end{array}$$

- By the Fundamental Property 2.30, $\sigma_0; \epsilon \vdash v \precsim^{\bullet} v : \forall\alpha.\alpha \to \alpha$.
- Construct $w_0 \in \mathrm{World}_{j+2}$ and $(\delta_1, \delta_2, \rho) \in D_{j+2}[\![\sigma_0]\!]w_0$ in the same manner as in the proof of Soundness (2.32) except that $w_0.\sigma_1 = w_0.\sigma_2 = \sigma$.
- Instantiating $\sigma; \epsilon \vdash v \precsim^{\bullet} v : \forall\alpha.\alpha \to \alpha$ then yields $(j+1, \lfloor w_0 \rfloor_{j+1}, v, v) \in V_n[\![\forall\alpha.\alpha \to \alpha]\!]\rho$
- By Wrapping II (2.37) and Inclusion (2.3), $(j+1, \lfloor w_0 \rfloor_{j+1}, \mathrm{Wr}^-(v), \mathrm{Wr}^-(v)) \in E_n^\circ[\![\forall\alpha.\alpha \to \alpha]\!]\rho$.
- Consequently, there exists $(j+1-j_1, w_1) \sqsupseteq (j+1, \lfloor w_0 \rfloor_{j+1})$ such that $\sigma; \mathrm{Wr}^-(v)\,\tau\,v' \hookrightarrow^* w_1.\sigma_2; (\Lambda\alpha.e_2)\,\tau\,v'$ with $w_1.\sigma_1 = \sigma_1$ and $(j+1-j_1, w_1, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^\circ[\![\forall\alpha.\alpha \to \alpha]\!]\rho$.
- Let $R := \{(k, w, v', v') \in \mathrm{Atom}_{j+1-j_1}[\sigma^*(\tau), \sigma^*(\tau)]\}$ and $r := (\rho^1(\tau), \rho^2(\tau), R)$, so $(\tau, \tau, r) \in T_{j+1-j_1}^\circ[\![\Omega]\!]w_1$.
- Instantiate $(j+1-j_1, w_1, \Lambda\alpha.e_1, \Lambda\alpha.e_2) \in V_n^\circ[\![\forall\alpha.\alpha \to \alpha]\!]\rho$ to get $(j+1-j_1-1, \lfloor w_1 \rfloor_{j+1-j_1-1}, e_1[\tau/\alpha], e_2[\tau/\alpha]) \in E_n^\circ[\![\alpha \to \alpha]\!]\rho, \alpha{\mapsto}r$.
- Consequently, there exists $(j+1-j_1-1-j_2, w_2) \sqsupseteq (j+1-j_1-1, \lfloor w_1 \rfloor_{j+1-j_1-1})$ such that $w_1.\sigma_2; e_2[\tau/\alpha]\,v' \hookrightarrow^* w_2.\sigma_2; (\lambda x.e_2')\,v'$ with $w_2.\sigma_1 = \sigma_2$ and $(j+1-j_1-1-j_2, w_2, \lambda x.e_1', \lambda x.e_2') \in V_n^\circ[\![\alpha \to \alpha]\!]\rho, \alpha{\mapsto}r$.
- Since $(j+1-j_1-1-j_2-1, \lfloor w_2 \rfloor, v', v') \in \lfloor R \rfloor = V_n^\circ[\![\alpha]\!]\rho, \alpha{\mapsto}r$, we get $(j+1-j_1-1-j_2-1, \lfloor w_2 \rfloor, e_1'[v'/x], e_2'[v'/x]) \in E_n^\circ[\![\alpha]\!]\rho, \alpha{\mapsto}r$.
- Consequently, there exists $(1, w_3) \sqsupseteq (j+1-j_1-1-j_2-1, \lfloor w_2 \rfloor)$ such that $w_2.\sigma_2; e_2'[v'/x] \hookrightarrow^* w_3.\sigma_2; v_2$ with $w_3.\sigma_1 = \sigma_3$ and $(1, w_3, v_1, v_2) \in V_n^\circ[\![\alpha]\!]\rho, \alpha{\mapsto}r = \lfloor R \rfloor$.
- Hence $v_1 = v_2 = v'$ by construction of R.

With the help of this lemma, we now prove the actual claim by contradiction.

- By the first part we know that if any such $\sigma; \mathrm{Wr}^-_{\forall\alpha.\alpha\to\alpha}(v)\,\tau\,v'$ terminates, then the result will indeed be v'.
- Suppose $\sigma_1; \mathrm{Wr}^-(v)\,\tau_1\,v_1 \hookrightarrow^j \sigma_1'; v_1$ but $\sigma_2; \mathrm{Wr}^-(v)\,\tau_2\,v_2 \uparrow$.
- By the Fundamental Property 2.30, $\sigma; \epsilon \vdash v \precsim^{\bullet} v : \forall\alpha.\alpha \to \alpha$.

- Construct $w_0 \in \mathrm{World}_{j+2}$ and $(\delta_1, \delta_2, \rho) \in D_{j+2}[\![\sigma_0]\!] w_0$ in the same manner as in the proof of Soundness (2.32) except that $w_0.\sigma_1 = \sigma_1$ and $w_0.\sigma_2 = \sigma_2$.
- Proceed as in the first part but pick $R := \{(k, w, v_1, v_2) \in \mathrm{Atom}_{j+1-j_1}[\sigma_1^*(\tau_1), \sigma_2^*(\tau_2)\}$ and $r := (\sigma_1^*(tau_1), \sigma_2^*(\tau_2), R)$, so $(\tau_1, \tau_2, r) \in T^\circ_{j+1-j_1}[\![\Omega]\!] w_1$.
- In the end we learn that $\sigma_2; \mathrm{Wr}^-(v)\, \tau_2\, v_2$ must terminate, too, contradicting the assumption.